PCA Module | Fall Term, 2001

Maths NYB — Cal II | Instructor: Dr. R.A.G. Seely

Models of Population Growth I:
An application of separable differential equations

1 A Game of Cat and Mouse

In the readings you have done, you have seen two models of population growth:

(1) Exponential growth, which is modeled by the differential equation $\dfrac{dP}{dt} = kP$, and

(2) Logistic growth, which is modeled by the differential equation $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right)$.

In this assignment we shall consider other ways to model the growth of a population of mice, based on the two models above but taking into consideration the effect of a cat moving into the neighbourhood. In class we saw how to solve these equations; in this assignment I want you to graph the direction fields for the equations to get some feel for *how* the solutions behave, and what reality they reflect.

1.1 To solve the questions posed here: ... a comment

This module will be somewhat easier to do if you use the software available to you. I have made this assignment available as a MAPLE worksheet, as well as providing a "hint" worksheet that ought to help you work through the assignment. Download that hint sheet, work through it to get a feel for how the program can do many of the tasks you are assigned, and then try the following questions using the same methods. Don't hand in your rough work however: hand in an edited version of this assignment, with just the questions, your solutions (together with sufficient explanatory text), and the actual MAPLE calculations that are needed for your answers. Your assignment mark will reflect the quality of the work, including presentation. (This paragraph is an example of text you ought **not** to hand in!)

Note: Don't try to learn MAPLE on the last morning, expecting to finish the assignment at the same time. This will take a little time — plan for that!! Also, keep in mind that you are using MAPLE to *help* you with the assignment — you still have to interpret your results to answer the questions I have asked you. Your MAPLE graphs and calculations are just the start of the work!

2 The Basic setting

A farmyard has a fertile mouse population. In this assignment we shall study this mouse population, making some simplifying assumptions based on observation and on analogy with the situations discussed in the readings you have done.

Let's suppose that on Monday morning (8am) we estimate the population to be 1600 mice. Suppose also to begin with that the growth of this population follows the logistic growth model.

Use the variables t for the time elapsed since 8am (in days), P for the population (in mice). Finally, imagine that at noon (the same Monday) the population was estimated at 1650 mice. Furthermore, suppose that observations of the mouse population in past years has suggested that the population has a maximum (carrying capacity) of about 6500 mice.

2.1 Logistic growth — Plotting a direction field by hand

To start with, we investigate what the logistic growth model predicts about the growth of the mouse population. Before using Maple to graph the setting we have, I want you to sketch a sample direction field by hand, so you can see that some behaviour can be guessed even before we have the plot. So, as a "toy model", let's look at the following simple logistic equation.

$$\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{10}\right)$$

Using the values $P = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$ and $t = 0, 1, 2, 3, 4, 5$, calculate the values of $\frac{dP}{dt}$ (note that this value does not in fact depend on t), and then plot a direction field for the equation. (Note that each horizontal row will have parallel direction arrows.) For what values of P is $\frac{dP}{dt} = 0$? What is the (obvious) connection with the equation $0.5P\left(1 - \frac{P}{10}\right) = 0$? Sketch (use another colour!) on your direction field the solution curves that correspond to initial values $P(0) = 1$ and $P(0) = 14$. To what value of P do these solution curves approach as $t \to \infty$? Finally, notice how much of this can be predicted just by looking at the quadratic expression $0.5P\left(1 - \frac{P}{10}\right)$ — you will need to notice such things when we look at the variations of this equation in dealing with our mouse population.

2.2 Logistic growth — Plotting a direction field with MAPLE

Next I want you to draw the direction field for the mouse population as given in "the basic setting", using Maple. So, using the data described in "the basic setting", set up and solve the differential equation for the mouse population. Be sure to evaluate any constant (*e.g.* k and K) that appears in the equation. Draw the direction field, and on it the graph of your solution.

2.3 Commentary

By analyzing the direction field, determine what would happen if the initial mouse population was larger than the carrying capacity of 6500 (*e.g.* if it were 8000 to start with at 8am, with the same value of k as calculated above)? (You can put the graph for this case on the same direction field.)

Use the graph to estimate the value of $\lim_{t\to\infty} P(t)$? What does this mean? How does this contrast with the exponential model? Of the two models, exponential and logistic, which is the more likely to be realistic in a setting like this? Why?

3 Enter the Cat

3.1 Linear hunting success rate

Every well-run farm has its cat, and this one is no exception. The model above has completely ignored this part of the situation, and now we shall try and fix that. Mug (the cat) is a good hunter, and so acts to reduce the rate at which the mouse population grows. The success of his hunting is directly proportional to the size of the mouse population, so this modifies the logistic equation to give the following differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) - hP$$

(h is a constant, Mug's *hunting success rate*). Use the values of k, K that you found *via* the logistic equation. Why is this a reasonable assumption? (Recall what these values represent, and explain why they are probably the same in this situation.) Use computer software to graph the direction field with your solution, comparing it to the previous solutions you obtained. You may suppose that Mug's success rate is 50 mice per day for a population of 1000 mice. (Hint: this means $h =$ 5%.)

What is the value of $\lim_{t\to\infty} P(t)$? Again, interpret this and contrast it with the previous models.

Try your calculations with some other values of h — *e.g.* h = 15%, 20%, 30%. How does this affect the direction fields? (Sketch them.) What effect does it have on the eventual outcome for the mouse population?

3.2 Other hunting models: quadratic and constant success rates

The hunting equation above is not the only way the cat's activities could be modelled; let's see what effects modifying it will have.

3.2.1 Quadratic success rate

The previous equation supposed that the success was proportional to the population, so *proportionally* speaking, Mug is as successful with small populations as he is with large ones. He may catch fewer mice, but the proportion of the mouse population won't change (5%). But it may very well be that as the population grows, hunting becomes easier for Mug, and indeed, not only does he catch more mice, but even proportionally speaking he catches a larger proportion of the population as the population grows. And conversely, if the population of mice were to shrink, his success might well shrink as well.

Modify the previous equation so that instead of a linear hunting success rate, we suppose a quadratic one:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) - hP^2$$

Suppose that k, K remain the same. Is this still justified? (Again, explain why in terms of that these numbers represent.) Is it reasonable to suppose that h is the same as before? (Why? Why not?) See if you can find a value of h that gives roughly the same limiting value as you got with h = 0.05 using the linear hunting equation above. Start with h = 0.00001.

Again, plot the direction field and graph, and compare it to the previous ones.

Do this with some other values of h (I suggest you try 0.00001, 0.00005, 0.0001, 0.001).

In each case, what is the value of $\lim_{t\to\infty} P(t)$? Interpret this and contrast with the previous models.

What if the initial population is much larger (say, 8000), or much smaller (say, 100)? Illustrate your reply with the direction field and solution graph for one value of h (*e.g.* 0.00001) and the two initial populations above.

3.2.2 Constant hunting success rate

We shall end with a final modification: the equations above implied that the larger the mouse population, the more mice Mug will catch. But it may very well be that Mug is satisfied with a certain number of mice per day, regardless of the number of available mice. He can only eat so many. Suppose he is well-enough fed at the farmhouse that he is satisfied with 10 mice per day ($h = 10$).

Modify the previous equation so that instead of a linear hunting success rate, we suppose a constant one:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) - h$$

Again, graph the direction field and solution, and compare it to the previous ones. Do this with some other values of h (100, 250). In each case, estimate $\lim_{t\to\infty} P(t)$? Interpret this and contrast with the previous models.

Note: After doing this assignment, you will be given some population data and asked to derive a likely model (*i.e.* differential equation) which is consistent with that data, and to then use this model to predict population behaviour. So, look over the *types* of answers, graphs, and so on that you may associate with each type of growth model.

You should also remember the connection between the limiting (equilibrium) solutions for the differential equations and the solutions to the corresponding quadratic equations.