Instructor: Dr. R.A.G. Seely | Assignment $4\frac{1}{2}$

Calculus III (Maths 201–DDB)

We start with a bit of theory: the essence of the "proof" of the second derivative test.

1. $\mathbf{u} = u\,\mathbf{i} + v\,\mathbf{j}$ is a unit vector, $f(x,y)$ a function with continuous second partial derivatives. Show that the second directional derivative of f in the direction $\mathbf{u}$ is

   $$f_{\mathbf{uu}} = u^2 f_{xx} + 2uv f_{xy} + v^2 f_{yy}$$

   Hint: $f_{\mathbf{u}} = \nabla(f)\cdot\mathbf{u}$, and so $f_{\mathbf{uu}} = \nabla(f_{\mathbf{u}})\cdot\mathbf{u}$ — now you just have to expand this and simplify.

2. If $f(x,y)$ has continuous second partial derivatives, satisfying the inequalities $f_{xx} > 0$, $f_{yy} > 0$, $f_{xx}f_{yy} - f_{xy}^2 > 0$, show that for any unit vector $\mathbf{u}$, $f_{\mathbf{uu}} > 0$.

   Hint: Let $\mathbf{u} = u\,\mathbf{i} + v\,\mathbf{j}$, and suppose $uv > 0$: expand the inequality $(u\sqrt{f_{xx}} - v\sqrt{f_{yy}}\,)^2 \geq 0$; notice that $\sqrt{A^2} = |A|$ and $|A| + A \geq 0$ for any A. If $uv < 0$, then expand $(u\sqrt{f_{xx}} + v\sqrt{f_{yy}}\,)^2 \geq 0$ and this time notice that $|A| - A \geq 0$ for any A.

   An alternate variant is to complete the square for the expression you found in question 1 for $f_{\mathbf{uu}}$.

   Notice that these two problems prove part of the 2DT: if f has an interior critical point at (x_0, y_0), and satisfies the conditions above near that point, then since all directional derivatives $f_{\mathbf{u}}(x_0,y_0) = 0$ and all second directional derivatives $f_{\mathbf{uu}}(x_0,y_0) > 0$, f must have a minimum at (x_0,y_0) (in every direction). The other parts of the 2DT are proven similarly.

3. Suppose $f(t)$ is a continuous function, and define $F(x,y) = \int_x^y f(t)\,dt$. What are $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$?

4. If $u = f(x,y)$ and $v = g(x,y)$, we define the *Jacobian* of u, v as

   $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

   (a) If $u = F(v)$ show that $\frac{\partial(u,v)}{\partial(x,y)} = 0$.
   Hint: Chain rule.

   (b) If $u = F(v,x)$, $v = g(x,y)$, and $\frac{\partial(u,v)}{\partial(x,y)} = 0$, show that $\frac{\partial F}{\partial x} = 0$ and so in effect $u = G(v)$ (for some function G).

   Hint: Show first that $\frac{\partial v}{\partial y}\,du - \frac{\partial u}{\partial y}\,dv = \frac{\partial(u,v)}{\partial(x,y)}\,dx$, according to the definition of differentials du, dv. So conclude $\frac{\partial v}{\partial y}\,du = \frac{\partial u}{\partial y}\,dv$. Then use the chain rule to calculate $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$. From that conclude that $du = \frac{\partial F}{\partial v}\,dv$, so arriving at a second equation for $\frac{\partial u}{\partial x}$: compare this with the one you got from the chain rule to conclude $\frac{\partial F}{\partial x} = 0$.