The errors of definitions multiply themselves according as the reckoning proceeds; and lead men into absurdities, which at last they see but cannot avoid, without reckoning anew from the beginning. —THOMAS HOBBES

Mathematicians are like lovers . . . Grant a mathematician the least principle, and he will draw from it a consequence which you must also grant him, and from this consequence another. —FONTENELLE

3 The Axiomatic Method

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I

1. EVOLUTION OF THE METHOD

IF the reader has at hand a copy of an elementary plane geometry, of a type frequently used in high schools, he may find two groupings of fundamental assumptions, one entitled "Axioms," the other entitled "Postulates." The intent of this grouping may be explained by such accompanying remarks as: "An *axiom* is a *self-evident truth*." "A *postulate* is a *geometrical fact* so simple and obvious that its validity may be assumed." The "axioms" themselves may contain such statements as: "The whole is greater than any of its parts." "The whole is the sum of its parts." "Things equal to the same thing are equal to one another." "Equals added to equals yield equals." It will be noted that such geometric terms as "point" or "line" do not occur in these statements; in some sense the axioms are intended to transcend geometry—to be "universal truths." In contrast, the "postulates" probably contain such statements as: "Through two distinct points one and only one straight line can be drawn." "A line can be extended indefinitely." "If L is a line and P is a point not on L, then through P there can be drawn one and only one line parallel to L." (Some so-called "definitions" of terms usually precede these statements.)

This grouping into "axioms" and "postulates" has its roots in antiquity. Thus we find in Aristotle (384–321 B.C.) the following viewpoint:[1]

> Every demonstrative science must start from indemonstrable principles; otherwise, the steps of demonstration would be endless. Of these indemonstrable principles some are (a) common to all sciences, others are (b) particular, or peculiar to the particular science; (a) the common principles are the *axioms*, most commonly illustrated by the axiom that, if equals be subtracted from equals, the remainders are equal. In (b) we have first the *genus* or subject-matter, the *existence* of which must be assumed.

[1] As summarized by T. L. Heath, *The Thirteen Books of Euclid's Elements*, Cambridge (England), 1908, p. 119. The reader is referred to this book for citations from Aristotle, Proclus, et al.

1.1. In Euclid's *Elements* (written about 300 B.C.), the two groups occur, respectively labeled "Common notions" and "Postulates." From these and a collection of definitions, Euclid deduced 465 propositions in a logical chain. Although the actual background for Euclid's work is not clear, apparently he did not originate this method of deducing logically from certain unproved propositions, given at the start, all the remaining propositions. As we have just noted, Aristotle, and probably other scholars of the period, had a well-conceived notion of the nature of a demonstrative science; and the logical deduction of mathematical propositions was common in Plato's Academy and perhaps among the Pythagoreans. Nevertheless, the influence of Euclid's work has been tremendous; probably no other document has had a greater influence on scientific thought. For example, modern high school geometries are usually modeled after Euclid's famous work (in England, Euclid is still used as a textbook), thus explaining the still common grouping into "axioms" and "postulates." Also the use in non-mathematical writings of such phrases as "It is axiomatic that . . . ," and "It is a fundamental postulate of . . . ," in the sense of something being "universal" or beyond opposition, is explained by this traditional use of the terms in mathematics.

The method featured in Euclid's work was employed by Archimedes (287–212 B.C.) in his two books which provided a foundation for the science of theoretical mechanics; in Book I of this treatise Archimedes proved 15 propositions from 7 postulates. Newton's famous *Principia*, first published in 1686, is organized as a deductive system in which the well-known laws of motion appear as unproved propositions, or postulates, given at the start. The treatment of analytic mechanics published by Lagrange in 1788 has been considered a masterpiece of logical perfection, moving from explicitly stated primary propositions to the other propositions of the system.

1.2. There exists a large literature devoted to the discussion of the *nature* of axioms and postulates and their philosophical background. Most of this is influenced by the fact that only within comparatively recent years have axioms and postulates been very generally employed in parts of mathematics other than geometry. Even though the method popularized by Euclid is acknowledged now as a fundamental part of the scientific method in every realm of human endeavor, our modern understanding of axioms and postulates, as well as our comprehension of deductive methods in general, has resulted to a great extent from studies in the field of geometry. And, since geometry was conceived to be an attempt to describe the actual physical space in which we live, there arose a conviction that axioms and postulates possessed a character of *logical necessity*. For example, Euclid's fifth postulate (the "parallel postulate") was "Let the following be postulated that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." [2] Proclus (A.D. 410–485) described vividly in his writings the controversy that was taking place in connection with this postulate even in his time; in fact, he argued in favor of the elimination "from our body of doctrine this merely plausible and unreasoned statement." [2] With the renewal of interest in Greek learning during the Renaissance, controversy in regard to the fifth postulate was renewed. Attempts were made to prove the "parallel postulate," often from logical—non-geometrical—principles alone. Surely if a statement is a "logical necessity" the assumption of its invalidity should lead to contradiction—such was the motivation of much of the work on the postulates of geometry. With the invention of non-euclidean geometries the futility of such attempts became clear.

1.3. The development of the non-euclidean geometries was evidence of a growing recognition of the independent nature of the fifth postulate; that is, this postulate cannot be demonstrated as a logical consequence of the other axioms and postulates in the euclidean system. By a suitable replacement of the fifth postulate, one may obtain the alternative and logically consistent geometry of Bolyai, Lobachevski, and Gauss in which the fifth postulate of Euclid fails to hold. In it appears, for example, the proposition that the sum of the interior angles of a triangle is less than two right angles. Riemann in 1854 developed another non-euclidean geometry, likewise composed of a non-contradictory collection of propositions, in which all lines are of finite length and the sum of the interior angles of a triangle is greater than two right angles.

The invention of the non-euclidean geometries was only part of the rapidly moving developments of the nineteenth century that were to lead to the acceptance of formal geometries apart from those that might be regarded as constituting definitive sciences of space or extension. Grassmann's *Ausdehnungslehre*, published in 1844 and a critical landmark during this era of changing ideas, was described by its author in these terms: "My *Ausdehnungslehre* is the abstract foundation for the doctrine of space, i.e., it is free from all spatial intuition, and is a purely mathematical discipline whose application to space yields the science of space. This latter science, since it refers to something given in nature (i.e., space), is no branch of mathematics, but is an application of mathematics to nature." [3] In explanation of Grassmann's concept of a formal science, Nagel writes: "Formal sciences are characterized by the fact that their sole principles of procedure are the rules of logic as well as by the further

[2] Quoted from T. L. Heath, op. cit., pp. 154–155, 203.

[3] As quoted by E. Nagel, "The formation of modern conceptions of formal logic in the development of geometry," *Osiris*, vol. 8 (1939), pp. 142–222, pp. 169, 172.

fact that their theorems are not 'about' some phase of the existing world but are 'about' whatever is *postulated* by thought." [3]

1.4. The idea expressed by Grassmann is essentially the one held at the present time; that is, a mathematical system called "geometry" is not necessarily a description of actual space. One must distinguish, of course, between the origin of a theory and the form to which it evolves. Geometry, like arithmetic, originated in things "practical," but to assert that any particular type of geometry is a description of physical space is to make a *physical* assertion, not a mathematical statement. In short, the modern viewpoint is that one must distinguish between mathematics and *applications* of mathematics.

A natural consequence of this change in viewpoint on the significance of a mathematical system was a re-examination of the nature of the basic, unproved propositions. It became clear, for instance, that the euclidean "common notion" that "the whole is greater than the part" has no more of an absolute character than the "parallel postulate" but is contingent upon the meaning of "greater than"; in fact, the proposition may even fail to hold, as in the theory of the infinite. Although there was much discussion as to whether the parallel postulate should be listed as a "postulate" or as a "common notion" (axiom), it was finally realized that neither had any more universality than the other and the distinction might as well be deleted.[4] Accordingly we find in the classical work of Hilbert on the foundations of geometry,[5] published in 1899, that only one name "axioms," is applied to the fundamental statements or assumptions, and that certain basic terms such as "point" and "line" are left completely undefined. To be sure, Hilbert made a grouping of his axioms—into five groups—but this pertained only to the technical character of the statements, and not to their relative status of "trueness" or "commonness."

1.5. Although this work of Hilbert has come to be regarded by many as the first to display the axiomatic method in its modern form, it should be recognized that similar ideas were appearing in works of his contemporaries. . . .

1.6. . . . Such studies as those of Pasch, Peano, Hilbert, and Pieri in euclidean geometry provided a tremendous impetus for investigations of possible formal organizations of the subject matter of this old discipline; these considerations, in turn, provided new understanding of mathematical systems in general and were partly responsible for the remarkable mathematical advances of the twentieth century. . . .

[4] For an excellent non-technical description of this "revolution" in thought, see E. T. Bell, *The Search for Truth*, Baltimore, 1934, chap. XIV.

[5] Hilbert, *Grundlagen der Geometrie*, Leipzig, 1899 (published in *Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen*); *The Foundations of Geometry*, Chicago, 1902. See also the seventh edition of *Grundlagen der Geometrie*. Leipzig, 1930.

It is noteworthy that these early studies in the field of geometry were revealing the great generality that was inherent in formal mathematical systems. Mathematics was evolving in a direction that was to compel the development of a method which could *encompass in a single framework* of undefined terms and basic statements concepts like *group* and *abstract space* that were appearing in seemingly unrelated branches of mathematics. . . . The economy of effort so achieved is one of the characteristic features of modern mathematics.

2. DESCRIPTION OF THE METHOD; THE UNDEFINED TERMS AND AXIOMS

As commonly used in mathematics today, the axiomatic method consists in setting forth certain basic statements about the concept (such as the geometry of the plane) to be studied, using certain undefined technical terms as well as the terms of classical logic. Usually no description of the meanings of the logical terms is given, and no rules are stated about their use or the methods allowable for proving theorems; perhaps these omissions form a weakness of the method.[6] The basic statements are called *axioms* (or, synonymously, *postulates*). It is assumed that in proving theorems from the axioms the rules of classical logic regarding contradictions and "excluded middle" may be employed; hence the "reductio ad absurdum" type of proof is in common use. The statements of both the axioms and the theorems proved from them are said to be *implied by* or *deduced from* the axioms. An example might be instructive:

2.1. Let us consider again the subject of plane geometry. It will be unnecessary to recall many details. We may perhaps assume, however, that the reader recalls from his high school course that *points* and *straight lines*, and such notions as that of *parallel* lines, were fundamental. Now, if we were going to set forth an axiomatic system for plane geometry in rigorous modern form, we would first of all select certain basic terms that we would leave undefined; perhaps "point" and "line" would be included here (the adjective "straight" can be omitted, since the *undefined* character of the term "line" enables us to choose to *mean* "straight line" in our thinking as well as in the later selection of statements for the axioms). Next we would scan the propositions of geometry and try to select certain basic ones with an eye to both their simplicity and their adequacy for proving the ones not selected; these we would call our *primary propositions* or *axioms*, to be left unproved in our system.

2.2. To be more explicit, let us proceed as though we were actually carry-

[6] We are not here describing the method as used in modern mathematical logic or the formalistic treatises of Hilbert and his followers, where the rules for operations with the basic symbols and formulas are (of necessity) set forth in the language of ordinary discourse.

ing out the above procedure; although we do not intend to give a complete system of axioms, a miniature sample of what the axioms and *secondary propositions* or *theorems* might be like, together with sample proofs of the latter, follows:

Undefined terms: Point; line.

Axiom 1. Every line is a collection of points.

Axiom 2. There exist at least two points.

Axiom 3. If p and q are points, then there exists one and only one line containing p and q.

Axiom 4. If L is a line, then there exists a point not on L.

Axiom 5. If L is a line, and p is a point not on L, then there exists one and only one line containing p that is parallel to L.

These axioms would not by any means suffice as a basis for proof of all the theorems of plane geometry, but they will be sufficient to prove a certain number of the theorems found in any organization of plane geometry. Their selection is motivated as follows: In the first place, the undefined terms "point" and "line" are to play a role like that of the variables in algebra. Thus, in the expression

$$x^2 - y^2 = (x - y)(x + y)$$

the x and y are undefined, in the sense that they may represent any individual numbers in a certain domain of numbers (as for instance the domain of ordinary integers). In the present instance, "point" may be any individual in a domain sufficiently delimited as to satisfy the statements set forth in the axioms. On the other hand, "line," as indicated in Axiom 1, has a range of values (= meanings) limited to certain collections of the individuals that are selected as "points." Thus Axiom 1 is designed to set up a *relationship* between the undefined entities *point* and *line*. It is *not* a *definition* of line, since (if the study is carried through) there will be other collections of points (circles, angles, etc.) that are not lines. Furthermore, it enables us, as we shall see presently, to define certain terms needed in the statements of the later axioms. Axiom 2 is the first step toward introducing lines into our geometry, and this is actually accomplished by adding Axiom 3. Before the latter can have meaning, however, we need the following formal definition:

2.3. Definition. If a point p is an element of the collection of points which constitutes a line L (cf. Axiom 1), then we say, variously, that *L contains p*, *p is on L*, or L is a line containing p.

Having stated Axioms 2 and 3, we would have that there exists a line in our geometry, but in order to have *plane* geometry and not merely a *line* or "one-dimensional" geometry we would have to say something to insure that not all points lie on a single line; Axiom 4 is designed to accomplish this. We would now imagine, intuitively (since we have a line L,

a point p not on that line, and also a line through p and each point q of L), that we have practically a plane; however, so far as euclidean geometry is concerned, we have not provided, in Axioms 1–4, for the *parallel* to L through p until we have stated Axiom 5. And of course Axiom 5 is not significant until we have the definition:

2.4. Definition. Two lines L_1 and L_2 are called *parallel* if there is no point which is on both L_1 and L_2. (We may also call L_1 "parallel to" L_2, or conversely.)

2.5. Let us denote the above set of five axioms, together with the undefined terms point, line, by Γ and call it the *axiom system* Γ.

(We shall also frequently use the term "axiom system" in a broader sense to include the theorems, etc., implied by the axioms.)

For future purposes we note two aspects of Γ, but we shall not go into these fully at this point: (1) In addition to the *geometrical* ("technical") undefined terms *point, line*, we have used *logical* ("universal") undefined terms such as *collection, there exist, one, every*, and *not*. (2) That Γ is far from being a set of axioms adequate for plane geometry may be shown as follows: Since *point* and *line* are left undefined, we are at liberty to consider possible meanings for them, subject of course to the restriction that we take into account the statements made in the axioms. If we have been educated in the American or English school systems, our reactions to these terms will no doubt immediately be specialized, our geometric experience in the schools having the upper hand in our response. But let us imagine that the terms are entirely unfamiliar, although the logical terms used in the axioms are not unfamiliar, so that we may consider other possible meanings for point and line. Unquestionably this will involve considerable experimentation before suitable meanings are found. For example, we might first try letting "point" mean book and "line" mean library; we know from the statement in Axiom 1 that a line is a collection of points, and libraries form one of the most familiar collections in our daily experience. We can imagine that we live in a city, C, which has two distinct libraries, and that by library we mean either one of the libraries of C, and by book any one of the books in these two libraries. Axiom 2 becomes a valid statement: "There exist at least two books." However, Axiom 3 fails, since, if p and q designate books in different libraries, then there is no library that contains *p and q*. However, before trying other meanings for point and line, we notice that Axioms 4 and 5 are valid, becoming, respectively, "If L is a library, then there exists a book not on (i.e., in) L," and "If L is a library and p a book not on (i.e., in) L, then there exists one and only one library containing p that is parallel to (has no books in common with) L." [7]

[7] In parentheses we have placed the terms commonly employed in connection with libraries and books that are indicated by our definition of "on" and "parallel."

Now, impressed by our failure to satisfy Axiom 3 on our first attempt at meanings for "point" and "line," we may, with an eye on Axiom 3, try to imagine a community, which we denote by Z, of people in which everyone belongs to some club, but in such a manner that, if p and q are two persons in Z, then there is one and only one club of which p and q are both members. In other words, we may try letting "point" mean a *person in Z* and "line" mean a *club in Z*, and imagine that the club situation in Z is such that the statement just made is valid, so that Axiom 3 is satisfied. We would then have no difficulty in seeing that Axioms 1, 2, and 4 are satisfied: "A club in Z is a collection of people in Z"; "There exist at least two people in Z"; etc. However, Axiom 5 becomes (with suitable change of wording to suit the new meanings): "If L is a club in Z, and p is a person in Z not in the club L, then there exists one and only one club in Z of which p is a member and which has no members in common with L." This is a statement which apparently makes a rather strong convention regarding the club situation in Z, and which may conceivably fail to apply; in any case, the stipulation that only one club have a given pair of persons as members can hardly be expected to suffice for Axiom 5! To clinch the matter suppose that Z is a "ghost" community, there being only three persons, whom we shall designate by a, b and c respectively, living in Z; and that as a result of certain circumstances each pair ab, bc, and ac shares a secret from the third member of the community, so that we may consider this bond between each two as forming them into a club ("secret society") excluding the third member. Now, with the meaning of point and line as before, we see that Axioms 1–4 hold but Axiom 5 does not hold.

Before rejecting the latter attempt as impossible, however, let us imagine that Z has *four* citizens: a, b, c, and d. And suppose that *each pair* of these people forms a club excluding the other two members of the community; that is, there are six club consisting of ab, ac, ad, bc, bd, and cd. Now *all* the axioms of Γ are satisfied with the meanings *person in Z* for "point" and *club in Z* for "line"! And we may then notice that we could arrive at a similar example by taking *any* collection Z of *four* things a, b, c, and d, and, by letting "point" mean a member or *element* of the collection Z, and "line" mean any pair of elements of Z, satisfy the statements embodied in the axioms of Γ.

2.6. Although we may experience no particular thrill at this discovery—may, rather, begin to feel that it is a rather trivial game we are playing in toying with possible meanings for the system Γ—we might conceivably be beguiled into seeking an answer to questions such as: How many "points" must a collection have in order to serve as the basis for an example satisfying the statements in Γ? For a given collection at hand, how many "points" must a "line" have in order to satisfy Γ? (For example, a "line" above could not have consisted of *three* persons in Z in the case where Z has exactly four citizens.) Furthermore, if we have already a general knowledge of, or experience with, plane geometry, the above example shows us that Γ is far from being a sufficient basis for euclidean geometry; certainly an adequate set of axioms for plane geometry would exclude the possibility of the geometry permitting a set of only *four* points satisfying all the axioms.

Before proceeding any further with this general discussion, however, let us notice how theorems would be proved from such a system as Γ.

3. DESCRIPTION OF THE METHOD; THE PROVING OF THEOREMS

Having set down a system, such as Γ for instance, we then proceed to see what statements are *implied*, or can be *proved* or *deduced from* the system. Contrary to the manner in which we proceeded in high school, when we brought in all kinds of propositions and assumptions not included in the fundamental terms and axioms (such as "breadth"; "a line has no breadth"), and even drew diagrams and pictures embodying properties that we promptly accepted as part of our equipment,[8] we take care to use only points and lines, and those relations and properties of points and lines that are given in the axioms. (Of course, after we have proved a statement, we may use it in later proofs instead of going back to the axioms and proving it all over again.) There is no objection to drawing diagrams, provided they are used only to aid in the reasoning process and do not trick us into making assumptions not implied by the axioms; indeed, the professional mathematician uses them constantly. . . .

3.1. Consider the following formal theorem and proof:

Theorem 1. *Every point is on at least two distinct lines.*

Proof. Let p denote any point. Since by Axiom 2 there exist at least two points, there must exist a point q distinct from p. And by Axiom 3 there exists a line L containing p and q. Furthermore, by Axiom 4 there exists a point r not on L, and (again by Axiom 3) a line K containing p and r.

Now by Axiom 1 every line is a collection of points. Hence, for two lines to be distinct (i.e., different), the two collections which constitute them must be different; or, what amounts to the same thing, one of them must contain a point that is not on the other. The lines L and K are distinct, then, because K contains the point r which is not on L. As p is on both L and K, the theorem is proved.

3.2. Now it will be noticed that we have used Axioms 1–4 in the proof,

[8] A classical example may be found in the well-known "proof" that all triangles are isosceles, which is based on a diagram that deceives the eye by placing a certain point *within* an angle instead of outside, where rigorous reasoning about the situation would place it. This may be found in J. W. Young, *Lectures on Fundamental Concepts of Algebra and Geometry*, New York, 1916, pp. 143–145.

but not Axiom 5. We could, then, go back to the example of the community Z, let "point" mean *person in Z* and "line" mean *pair of persons in Z*, rephrase Axioms 1–4 in these terms, and carry through the proof of Theorem 1 in these terms. That is, Theorem 1 is a "true" statement about any example, such as Z, which satisfies the statements embodied in Axioms 1–4 of Γ. In proving Theorem 1, then, *we have in one step proved many different statements about many different examples*, namely, the statements corresponding to Theorem 1 as they appear in the different examples that satisfy Axioms 1–4 of Γ. This [is an important] aspect of the "economy" achieved in using the axiomatic method. . . . If, because of some diagram or other aid to thought, we had used some property of point or line not stated in Axioms 1–4, we could not expect to make the above assertions, and the "economy" cited would be lost!

Note, too, that Theorem 1 will remain valid in any axiom system (such as Γ) that *contains* the undefined terms point and line as well as Axioms 1–4. In particular, it is valid for euclidean plane geometry, which is only one of the possible geometries embodying these four axioms, and which, as we stated before, would require many more axioms than those stated above.

3.3. Now consider the following statement, which we call a *corollary* of Theorem 1:

Corollary. *Every line contains at least one point.*

3.4. Before considering a proof, we hasten to meet an objection which the "uninitiated" might make at this point; to wit, since Axiom 1 explicitly states that a line is a collection of points, *of course* every line contains at least one point, so why should this be repeated as a corollary of Theorem 1? This is not a trivial matter, and it leads directly to a question which causes considerable concern in modern mathematics, namely, what is meant by *collection*? We said above that "collection" is an undefined logical term, and as such we took it for granted that its use is universally understood and employed, just as the word "the" is universally understood and used by anyone familiar with the English language. But now we find ourselves almost immediately in need of explaining the use of the term in the above corollary.

However, there is nothing so very astonishing about this if we reflect that, whenever we try to make very precise a term in ordinary use, it is usually necessary to adopt certain conventions. For example, such terms as *vegetable, fruit, animal* are commonly "understood" and used by anyone who habitually uses the English language, but, when we come to apply them to certain special objects, it is frequently necessary to agree on some convention; as, for example, that a certain type of living substance shall be called "animal" rather than "fish" (e.g., *whale*). So, for instance, we may want to make the convention that, if person A wishes to talk about "the collection of all coins in B's pockets," he may do so even though person B is literally penniless! In other words, no matter whether there actually are coins in B's pockets or not, the collection of all such coins is to be regarded as an existing entity; we call the collection *empty* if B has no coins. (In case B is penniless, we may also talk about "the coin in B's pocket," but in this case there is no existing entity to which the phrase refers.) And this is the convention that is generally agreed on throughout mathematics and logic, namely, that a collection may "exist," as in the case of the collection of all coins in B's pockets, even though it is empty. . . .

3.5. Proof of corollary to Theorem 1. There exists a point p by Axiom 2, and by Theorem 1 there exist two distinct lines L_1 and L_2 containing p.

Now, if there exists a line L that contains no points, then both L_1 and L_2 are parallel to L (by definition). As this would stand in contradiction to Axiom 5, it follows that there cannot exist such a line L.

3.6. A statement "stronger" than the above corollary is embodied in the next theorem:

Theorem 2. *Every line contains at least two points.*

Proof. Let L be any line. By the above corollary, L contains a point p. To show p not the only point on L, we shall use a "proof by contradiction." Suppose p is the only point that L contains. By Theorem 1 there is another line L_1 containing p. Now L_1 must contain at least one other point, q; for otherwise L and L_1 would each contain only p, hence be the same collection of points and ergo the same line (Axiom 1). By Axiom 4 there is a point x which is not on L_1, and by Axiom 5 there is a line L_2 containing x and parallel to L_1. But both L and L_1 are lines containing p and parallel to L_2, in violation of Axiom 5. We must conclude, then, that the supposition that p is the only point on L cannot hold and hence that L contains at least two points.

Now, since by Theorem 2 every line contains at least two points, and since by Axiom 3 two given points can lie simultaneously on only one line, we can state:

Corollary (to Theorem 2). *Every line is completely determined by any two of its points that are distinct.*

3.7. Theorem 3. *There exist at least four distinct points.*

Proof. By Axiom 2 there exist at least two distinct points p and q. By Axiom 3 there exists a line L containing p and q, and by Axiom 4 there exists a point x not on L. By Axiom 5 there exists a line L_1 containing x and parallel to L, and by Theorem 2 L_1 contains at least two distinct points (cf. Definition 2.4).

3.8. Theorem 4. *There exist at least six distinct lines.*

Before proving Theorem 4, we perhaps need to make sure that the meaning of another one of our "common" terms is agreed upon, namely the word "distinct." As we are using the term, two collections are distinct if they are not the *same*. Thus, the lines L and L_1 which figure in the proof of Theorem 2 are distinct, although under the supposition made there L_1 *contains* L, for they are not the *same* line (L_1 contains q and L does not).

Proof of Theorem 4. We proceed, as in the proof of Theorem 3, to obtain the line L containing the points p and q, and the line L_1 parallel to L containing two distinct points (Theorem 2) x and y. By Axiom 3 there exist lines K and K_1 determined respectively by the pairs (p, x), (q, y). Now the point q is not on K, else by Axiom 3 K and L would be the same line (which is impossible since x is not on L). Also, y is not on K, else K and L_1 would be the same line. Similarly, p is not on K_1 and x is not on K_1. Now there also exist lines M and M_1 determined respectively by the pairs (p, y), (q, x); and we can show that q is not on M, x is not on M, p is not on M_1, and y is not on M_1. It follows that no two of the lines L, L_1, K, K_1, M, M_1 are the same.

4. COMMENT ON THE ABOVE THEOREMS AND PROOFS

If the reader has followed the proofs given above, he has probably resorted to the use of figures by this time! This would be quite natural, since in high school geometry he used figures; and they help to keep the various symbols $(L, p, q, \cdots)$ and their significance in mind. However, as we stated above, no special meanings have been assigned to "point" and "line," and consequently the above proofs should, and do, hold just as well if the reader uses coins for "points" and pairs of coins for "lines." As a matter of fact, if *any* collection of four objects is employed, "point" meaning any object of the collection and "line" any pair of the objects, then the reader may follow the above proofs with these meanings in mind.

Of course, the theorems we have stated in the preceding sections are not by any means all the theorems that we might state. For example, we can show that any collection of objects satisfying the axioms of the system Γ must, if not infinite as in ordinary geometry, satisfy certain conditions regarding the number of points (there cannot be just 5 points in the collection, for instance), and that there must be a relation between the number of lines and the number of points in the collection. In fact, we can continue the above study to a surprising extent; we could hardly expect to reach a point where we could confidently assert that no more theorems could be proved. It is not our intention to extend the number of theorems, however, since we believe that we have already obtained

enough theorems and proofs to serve as specimens for our later purposes.

4.1. As a useful terminology in what follows, let us agree that, when we use the term "statement" in connection with an axiom system Σ, we shall mean a sentence phrased, or phrasable, in the undefined terms and universal terms of Σ; such a statement may be called a *Σ-statement*. Thus the axioms of Γ are Γ-statements (Axiom 5 contains the word "parallel," but this is "phrasable" in the undefined terms and universal terms), as are also the theorems.

4.2. In conformance with the conventions made in Section 2, we shall say that an axiom system Σ *implies* a statement S if S follows by logical argument, such as used above, from Σ. In particular, each axiom is itself implied, trivially. We shall also say S is *logically deducible* from Σ if Σ implies S.

4.3. In the course of our work above we had to pause in two instances to explain the conventions we were making in regard to the use of two words commonly used in ordinary discourse, namely "collection" and "distinct." These words were left undefined, to be sure, in the sense that they are supposedly universally understood non-technical terms; but, as we discovered, not so "universal" but that it was felt advisable to give some conventions we were making in regard to their use here! On the other hand, the words "point" and "line" we left strictly undefined, saying that any meaning whatsoever could be assigned to them as long as these meanings were consistent with the statements embodied in the axioms. We saw that the "collection = library," "point = book" meanings were not permissible, but that, if C is any collection of four objects, then "point = object of C," "line = pair of objects of C" are permissible meanings. The terms "point," "line," "parallel," etc., we may call *technical* terms of the system, the terms "point" and "line" being the *undefined technical terms*. The terms "collection," "distinct" might be called *universal terms* or *logical terms*.

Other examples of universal terms in Γ are "exist" (in Axiom 2), "one" (Axiom 3), "two" (Theorem 1), "four" (Theorem 3), "six" (Theorem 4), "and" (Axiom 3), "or" (Definition 1), "not" (Axiom 4), and "every" (Theorem 1). However, if we were setting up an axiom system for the elementary arithmetic of integers ($1 + 2 = 3$, $2 \times 2 = 4$, etc.), we might use a term like "one" as an undefined technical term. Thus the same term may have different roles in different axiom systems! . . . As the term "exist" is used above, it is chiefly *permissive* so far as proofs are concerned, and *stipulative* for examples; thus in the proof of Theorem 3 we were permitted to introduce the line L_1 by virtue of Axiom 5, and the example of the "ghost community" containing only three persons failed because it could not meet the stipulation concerning the existence of a certain line parallel to another line which is made in Axiom 5.

5. SOURCE OF THE AXIOMS

Let us consider more fully the *source of the statements* embodied in the axioms. We chose axioms for *geometry* in our example Γ since we felt that we could assume the reader had studied some elementary geometry in high school. That is, we were careful to pick a subject already familiar! The undefined technical terms "point" and "line" already have a meaning of some sort for us. And . . . this is the usual way in which axioms are obtained; *they are statements about some concept with which we already have some familiarity*. Thus, if we are already familiar with arithmetic, we might begin to set down axioms for arithmetic. Of course the method is not restricted to mathematics. If we are familiar with some field such as physics, philosophy, chemistry, zoology, economics, for instance, we might choose to set down some axioms for it, or a portion of it, and see what theorems we might logically deduce from them.[9] We may say, then, that an *axiom*, as used in the modern way, is *a statement which seems to hold for an underlying concept*, an *axiom system* being *a collection of such statements about the concept*.

Thus, in practice, the concept comes first, the axioms later. Theoretically this is not necessary, of course. Thus, we may say "Let us take as undefined terms *aba* and *daba*, and set down some axioms in these and universal logical terms." With no concept in mind, it is difficult to think of anything to say! That is, unless we first *give some meanings* to "aba" and "daba"—that is, introduce some *concept* to talk about—it is difficult to find anything to say at all. And, if we finally do make some statements without first fitting a suitable concept to "aba" and "daba," we shall, very likely, make statements that contradict one another! The underlying concept is not only a source of the axioms, but it also guides us to *consistency* (about which we shall speak directly).

Thus, we select the concept; then we select the terms that are to be left undefined and the statements that are to form our axioms; and finally we prove theorems as we did above. This is a simplification of the process, to be sure, but in a general way it describes the method. It is to be noted how the procedure, as so formulated, differs from the classical use of the method. In the classical use the axioms were regarded as absolute truths—absolutely true statements about material space—and as having a certain character of necessity. To have stated the parallel axiom, Axiom 5 above, was to have stated something "obviously true," something one took for granted if one had thought about the character of the space in which he lived. It would have been inconceivable before the nineteenth century to state an axiom such as "If L is a line and p a point not on L, then there

[9] As an example in genetics and embryology, see J. H. Woodger, *The Axiomatic Method in Biology*, Cambridge (England), 1937.

exist at least two distinct lines containing p and parallel to L." To have in mathematics, simultaneously, two axiom systems Γ_1 and Γ_2 with axioms in Γ_1 denying axioms in Γ_2 as is the case in mathematics today with the euclidean and non-euclidean geometries, would also have been inconceivable! But, if we take the point of view that an axiom is only a statement about some concept,[10] so that axioms contradicting one another in different systems only express basic differences in the concepts from which they were derived, we see that no fundamental difficulty exists. What is important is that axioms in the *same* system should not contradict one another. This brings us to the point where we should discuss consistency and other characteristics of an axiom system.

5.1. REMARK

The derivation of an axiom system for non-euclidean geometry from axioms for euclidean geometry, using the device of replacing the parallel axiom by one of its denials, is an example of another manner in which new axiom systems may be obtained. In general, we may select a given axiom system and change one or more of the axioms therein in suitable manner to derive a new axiom system.

II

ANALYSIS OF THE AXIOMATIC METHOD

[When the undefined terms and primary propositions or axioms of a system have been selected at least three relevant questions suggest themselves: (1) Is the system suited to the purposes for which it was set up? (2) Are the axioms truly independent, i.e., are any of them provable from the others (in which case they should perhaps be deleted from the system and transferred to the body of theorems to be proved)? (3) Does the system imply any contradictory theorems (if so, this defect must be eliminated if the theorems are to be relied on)? Of these questions, the third, relating to contradictoriness, is by far the most fundamental and critical. In the selection below, a continuation of the preceding discussion, I have excerpted Wilder's analysis of the consistency and independence of an axiom system. ED.]

1. CONSISTENCY OF AN AXIOM SYSTEM

From a logical point of view we can make the following definition:

1.1. Definition. An axiom system Σ is called *consistent* if contradictory statements are not implied by Σ.

Now this definition gives rise to certain questions and criticisms. In the first place, given an axiom system Σ, *how are we going to tell whether*

[10] It is only in this sense—that an axiom is a statement *true* of some concept—that the word "true" can be used of an axiom.

it is consistent or not? Conceivably we might prove two theorems from Σ which contradict one another, and hence conclude that Σ is *not* consistent.

For example, if we added to the system Γ, discussed above, the new axiom, "There exist at most three points," it would become apparent, as soon as Theorem 3 of Γ was proved, that the new system of axioms is not consistent.

But, supposing that this does not happen, are we going to conclude that Σ is consistent? How can we tell that, if we continued stating and proving theorems, we might not ultimately arrive at contradictory statements and hence *inconsistency*? We remarked about the system Γ that we could hardly expect to reach a point where we could say with confidence that no more theorems could be stated. And, unless we could have all possible theorems in front of our eyes, capable of being scanned for contradictions, how could we assert that the system is consistent? We are immediately faced with the problem: Is there any *procedure for proving a system of axioms consistent?* And, if so, on what basis does the proof rest, since the proof cannot be conducted *within* the system as in the case of the theorems of the system?

Another difficulty would arise from the fact that it might be very hard to recognize that a contradiction is implied, when such is the case. There are examples in mathematical literature of cases where considerable material was published concerning systems which later were found to be inconsistent. Until someone suspected the inconsistency and set out to prove it, or (as in some cases) stumbled upon it by chance, the systems seemed quite valid and worth while. It can also happen, for example, that the theorems become so numerous and complicated that we fail to detect a pair of contradictory ones. For example, although two theorems might really be of the form "S" and "not S" respectively, because of the manner in which they are stated it might escape our attention that they contradict one another. In short, the usefulness of the above definition is limited by our ability to recognize a contradiction even when it is staring us in the face, so to speak.

The former objection, that hinging upon the probable impossibility of setting down all theorems implied by the system, is the more serious from the point of view of the working mathematician. And as a consequence the mathematician usually resorts to the procedure described below:

Let us make the definition:

1.2. Definition. If Σ is an axiom system, then an *interpretation* of Σ is the assignment of meanings to the undefined technical terms of Σ in such a way that the axioms become true statements for all values of the variables (such as p and q of Axiom 3, I 2.2, for instance).

This definition requires some explanation. First, as an example we can

cite the system Γ above and the meanings "point" = any one of a collection of four coins and "line" = any pair of coins in this collection. The axioms now become statements about the collection of coins and are easily seen to be true thereof. Hence this assignment of meanings is an interpretation of Γ. As the axioms stand, with "point" and "line" having no assigned meanings, they cannot be called either true or not true. (Similarly, we cannot speak of the expression "$x^2 - y^2 = (x - y)(x + y)$" as being either true or false until *meanings*, such as "x and y are integers," are assigned.) But, with the meanings assigned above, they are true statements about a "meaningful" concept. As a rule, we shall use the word "model" to denote the *result* of the assignment of meanings to the undefined terms. Thus the collection of four coins, considered a collection of points and lines according to the meanings assigned above, is a model of Γ. Generally, if an interpretation I is made of an axiom system, we shall denote the model resulting from I by $\mathfrak{M}$(I).

For some models of an axiom system Σ, certain axioms of Σ may be *vacuously satisfied*. That is, axioms of the form "If . . . , then . . . ," such as Axiom 3 of Γ, which we might call "conditional axioms," may be true as interpreted only because the conditional "If . . ." part is not fulfilled by the model.

Suppose, for example, we delete Axioms 2 and 4 from Γ and denote the resulting system by Γ′. Then a collection of coins containing just one coin is a model for Γ′, if we interpret "point" to mean coin and "line" to mean a collection containing just one coin. For in this model the "If . . ." parts of Axioms 3 and 5 are not fulfilled. (Note that, for Axiom 3 to be false of a model $\mathfrak{M}$, there must be *two* points p and q in $\mathfrak{M}$ such that either no line of $\mathfrak{M}$ contains p and q or more than one line of $\mathfrak{M}$ contains p and q.)

This may be better illustrated, perhaps, by the following digression: Suppose boy A tells girl B, "If it happens that the sun shines Sunday, then I will take you boating." And let us suppose that on Sunday it rains all day, the sun not once peeping out between the clouds. Then, no matter whether A takes B boating or not, it cannot be asserted that he made her a false promise. For his promise to have been false, (1) the sun must have shone Sunday and (2) A must not have taken B boating. And, in general, for a statement of the form "If . . . , then . . ." to be false, the "If . . ." condition must be fulfilled and the "then . . ." not be fulfilled.

Now we did not have in mind a collection of four coins when we set down the axioms of Γ. We were thinking of something entirely different, namely euclidean geometry as we knew it in high school. "Point" had for us then an entirely different meaning—something "without length, breadth, or thickness"; and "line" meant a "straight" line that had "length, but no breadth or thickness." Do not these meanings also yield a model of Γ—what we might perhaps call an "ideal" model? We may admit that this is so, and we resort frequently in mathematics to such ideal models—always, of course, when it happens that every collection of objects serving as a model must of necessity be infinite in number. (Such is the case, for instance, when we have enough axioms in a geometry to insure an infinite number of lines.) We return to this discussion later (see 2.3); at present, let us go on to the so-called "working definition" of consistency:

1.3. Definition. An axiom system Σ is *satisfiable* if there exists an interpretation of Σ.

Now what is the relation between the two definitions 1.1 and 1.3? What we actually want of any axiom system is that it be consistent in the sense of 1.1. But we saw that 1.1 was not a practicable definition except in cases where contradictory statements are actually found to be implied by the system and inconsistency is thus recognized. Where a system is consistent, we are usually unable to tell the fact from 1.1. But, as in the case of the four-coin interpretation of Γ above, we have a very simple test showing "satisfiability" in the sense of 1.3. Does this imply consistency in the sense of 1.1? The mathematician and the logician take the point of view that it does, and, in order to explain why, we have to go into the domain of logic for a few moments.

2. THE PROOF OF CONSISTENCY OF AN AXIOM SYSTEM

The Law of Contradiction and the Law of the Excluded Middle.

First let us recall two basic "laws" of classical (i.e., Aristotelian) logic, namely the Law of Contradiction and the Law of the Excluded Middle; the latter is also called the Law of the Excluded Third ("tertium non datur"). These are frequently, and loosely, described as follows: If S is any statement, then the Law of Contradiction states that S and a contradiction (i.e., any denial) of S cannot both hold. And the Law of the Excluded Middle states that either S holds or a denial of S holds. For example, let S be the statement "Today is Tuesday." The Law of Contradiction certainly holds here, for today cannot be *both* Tuesday and Wednesday, for example. And the Law of the Excluded Middle states that either today is Tuesday or it is not Tuesday.

But "things are not so simple as they seem" here. Unless one limits himself to a specified point on the earth (or parallel of longitude), it *can* be both Tuesday and Wednesday at the same time! And, without including in S such a geographical provision, the statement "Either today is Tuesday or it is not Tuesday" can hardly be accepted. As a matter of fact, whenever such statements are made there usually exists a tacit understanding between speaker and listener that their locale at the time is the place being referred to.

Or consider the statement "The king of the United States wears bow ties." Does the Law of the Excluded Middle hold here? Or let S be the statement "All triangles are green."

The upshot of this is merely that, although these "laws" are called "universally valid," some sort of qualifications have to be made with regard to their applicability in order for them to have validity. In so far as axiomatic systems are concerned, the problem is not so great, since we can restrict our use of the term "statement" to the convention already made in section I 4.1 ("Σ-statement"). And this will be our understanding from now on.

2.2. As soon as an interpretation of a system Σ is made, the statements of the system become statements about the resulting model. Let us assume the following, which may be considered *basic principles of applied logic.*

2.2.1. All statements implied by an axiom system Σ hold true for all models of Σ;

2.2.2. The Law of Contradiction holds for all statements about a model of an axiom system Σ, provided they are Σ-statements whose technical terms have the meanings given in the interpretation. We can make this clearer and more precise by introducing the notion of an I-Σ-statement:

2.2.3. If Σ is an axiom system and I denotes an interpretation of Σ, then the result of assigning to the technical terms in a Σ-statement their meanings in I will be called an I-Σ-statement.

Then 2.2.1 and 2.2.2 become respectively:

2.2.1. Every I-Σ-statement, such that the corresponding Σ-statement is implied by Σ, holds true for $\mathfrak{M}$(I) (cf. 1.2);

2.2.2. Contradictory I-Σ-statements cannot both hold true of $\mathfrak{M}$(I).

Under the assumption that 2.2.1 and 2.2.2 hold, satisfiability implies consistency. For, if an axiom system Σ implies two contradictory Σ-statements, then by 2.2.1 these statements as I-Σ-statements hold true for the model $\mathfrak{M}$(I); but the latter is impossible by 2.2.2. Hence we must conclude that, if 2.2.1 and 2.2.2 are valid, then the existence of an interpretation for an axiom system Σ guarantees the consistency of Σ in the sense of 1.1. And this is the basis for the "working definition" 1.3. For example, the existence of the "four-coin interpretation" of the system Γ above guarantees the consistency of Γ if we grant 2.2.1 and 2.2.2.

The reader will have noticed that we have not proved that consistency in the sense of 1.1 implies satisfiability. To go into this question would be impractical, since it would necessitate going into detail concerning formal logical systems and is too complicated to describe here.

2.3. In Section 1 we used the term "ideal" model, by way of contrast to such models as that of the four coins for Γ; the latter might be termed a "concrete" model. It was pointed out that, whenever an axiom system Σ requires an infinite collection in each of its models, then of necessity the models are "ideal."

This raises not only the question as to how reliable are "ideal" models, but also the question as to what constitutes an *allowable* model. What we should like, of course, is a criterion which would allow only models that satisfy assumptions 2.2.1, 2.2.2, and especially the latter. If there is any danger that an ideal model may require such a degree of abstraction that it harbors contradictions in violation of 2.2.2, then clearly the use of models is no general guarantee of consistency in spite of what we have said above.

Further light can be shed on this matter by a consideration of well-known examples. It is not an uncommon practice, for instance, to obtain a model of an axiom system Σ in another branch of mathematics—even in a branch of mathematics that is, in its turn, based on an axiom system Σ'. How valid are such models? Do they necessarily satisfy 2.2.2? For example, to establish the consistency of a non-euclidean geometry we give a model of it in euclidean geometry. (See Richardson [*Fundamentals of Mathematics*, New York, 1941, pp. 418–19] for instance.) But suppose that the euclidean geometry harbors contradictions; what then? Evidently all we can conclude here is that, *if* euclidean geometry is consistent, then so is the non-euclidean geometry whose model we have set up in the euclidean framework.

We are forced to admit that in such cases we have no absolute test for consistency, but only what we may call a *relative consistency proof*. The axiom system Σ' may be one in whose consistency we have great confidence, and then we may feel that we achieve a high degree of plausibility for consistency, but in the final analysis we have to admit that we are not *sure* of it.[11]. . .

3. INDEPENDENCE OF AXIOMS

Earlier we mentioned "independence" of axioms. By "independence" we mean essentially that we are "not saying too much" in stating our axioms. For example, if to the five axioms of the system Γ (I 2.2) we added a sixth axiom stating "There exist at least four points," we would provide no new information inasmuch as the axiom is already implied by Γ (see Theorem 3 of I 3.7). Of course the addition of such an axiom would not destroy the property of consistency inherent in Γ.

3.1. In order to state a formal definition of independence, let Σ denote an axiom system and let A denote one of the axioms of Σ. Let us denote some denial of A by $\sim$A, and let $\Sigma - A$ denote the system Σ with A deleted. If S is any Σ-statement, $\Sigma + S$ will mean the axiom system containing the axioms of Σ and the statement S as a new axiom. Then we define:

3.1.1. Definition. If Σ is an axiom system and A is one of the axioms of Σ, then A is called *independent* in Σ, or an independent axiom of Σ, if both Σ and the axiom system $(\Sigma - A) + \sim A$ are satisfiable.

3.2. Just which of the many forms of $\sim$A is used is immaterial. Thus Axiom 5 is independent in Γ (I 2.2) if Γ is satisfiable and if the first four axioms of Γ togther with a "non-euclidean" form of the axiom constitute a satisfiable system. For example, for a denial of Axiom 5 take the statement: "There exist a line *L* and point *p* not on *L*, such that there does not exist a line containing *p* and parallel to *L*." To show that the system Γ with Axiom 5 replaced by this statement forms a satisfiable system, let us take a collection of *three* coins, let "point" mean a coin of this collection, and "line" mean any pair of points of this collection. Then we have an interpretation of the new system, showing it to be satisfiable. We have already ascertained (2.2) that Γ is satisfiable, and so we conclude that Axiom 5 is independent in Γ.

3.3. The reader will probably gather by this time that the reason for specifying the satisfiability of Σ, in Definition 3.1.1, is to insure that some $\sim$A is not a *necessary* consequence of the axioms of $\Sigma - A$; for, if it were, we would not wish to call A "independent." And, as the definition is phrased, it insures that neither A nor any denial, $\sim$A, of A is implied by the system $\Sigma - A$, so that the addition of A to $\Sigma - A$ is really the supplying of new information.

3.4. Actually, however, we do not place the same emphasis on independence as we do on consistency. Consistency is *always* desired, but there may be cases where independence is *not* desired. . . . Generally speaking, of course, it is preferable to have all axioms independent; but, if some axiom turns out not to be independent, the system is not invalidated.

As a matter of fact, some well-known and important axiom systems, when first published, contained axioms that were not independent (a fact unknown at the time to the authors, of course). An example of this is the original formulation of the set of axioms for geometry given by Hilbert in 1899. This set of axioms contained two axioms which were later discovered to be implied by the other axioms.[12] This in no way invalidated the system; it was only necessary to change the axioms to theorems (supplying the proofs of the latter, of course). . . .

[11] In one well-known case, the system Σ' is a *subsystem* of Σ; viz., the Gödel proof (Gödel, *The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the Axioms of Set Theory*, Princeton, 1940) of the relative consistency of the axiom of choice when adjoined to the set theory axioms.

[12] See, for example, E. H. Moore, "On the projective axioms of geometry," *Trans. Amer. Math. Soc.*, vol. 3 (1902), pp. 142–158.