MATHEMATICAL LOGIC - Fitch-style Natural Deduction

Conjunction introduction (∧I)

```
⋮ | ⋮
m | A
⋮ | ⋮
n | B
⋮ | ⋮
p | A ∧ B    ∧I, m, n
```

```
⋮ | ⋮
m | B
⋮ | ⋮
n | A
⋮ | ⋮
p | A ∧ B    ∧I, m, n
```

Conjunction elimination (∧E)

```
⋮ | ⋮
m | A ∧ B
⋮ | ⋮
n | A        ∧E, m
```

```
⋮ | ⋮
m | A ∧ B
⋮ | ⋮
n | B        ∧E, m
```

Disjunction Introduction (∨I)

```
⋮ | ⋮
m | A
⋮ | ⋮
n | A ∨ B    ∨I, m
```

```
⋮ | ⋮
m | B
⋮ | ⋮
n | A ∨ B    ∨I, m
```

Disjunction Elimination (∨E)

```
⋮     | ⋮
m     | A ∨ B
m + 1 | | A
      | |----
⋮     | | ⋮
n     | | C
n + 1 | | B
      | |----
⋮     | | ⋮
p     | | C
p + 1 | C        ∨E, m, (m + 1)–n, (n + 1)–p
```

Implication Introduction (⇒I)

```
⋮     | ⋮
m     | | A
      | |----
⋮     | | ⋮
n     | | B
n + 1 | A ⇒ B    ⇒I, m–n
```

Implication Elimination (⇒E)

```
⋮ | ⋮
m | A
⋮ | ⋮
n | A ⇒ B
⋮ | ⋮
p | B        ⇒E, m, n
```

```
⋮ | ⋮
m | A ⇒ B
⋮ | ⋮
n | A
⋮ | ⋮
p | B        ⇒E, m, n
```

Negation Introduction (¬I)

```
⋮     | ⋮
m     | | A
      | |----
⋮     | | ⋮
n     | | ⊥
n + 1 | ¬A       ¬I, m–n
```

Negation Elimination (¬E)

```
⋮ | ⋮
m | A
⋮ | ⋮
n | ¬A
⋮ | ⋮
p | ⊥        ¬E, m, n
```

```
⋮ | ⋮
m | ¬A
⋮ | ⋮
n | A
⋮ | ⋮
p | ⊥        ¬E, m, n
```

Contradiction Elimination (⊥E)

```
⋮ | ⋮
m | ⊥
⋮ | ⋮
n | C    ⊥E, m
```

Double negation elimination (¬¬E)

```
⋮ | ⋮
m | ¬¬A
⋮ | ⋮
n | A        ¬¬E, m
```

Repetition (R)

```
m | A
⋮ | ⋮
⋮ | ⋯ | ⋮
n |   | A    R, m
```

Forall-introduction (∀I)

```
⋮     | ⋮
m     | u | ⋮
⋮     |   | ⋮
n     |   | A(u)
n + 1 | ∀x A(x)    ∀I, m–n
```

Forall-elimination (∀E)

```
⋮ | ⋮
m | ∀x A(x)
⋮ | ⋮
n | A(t)       ∀E, m
```

Exists-Introduction (∃I)

```
⋮ | ⋮
m | A(t)
⋮ | ⋮
n | ∃x A(x)    ∃I, m
```

Exists-Elimination (∃E)

```
⋮     | ⋮
p     | ∃x A(x)
m     | u | A(u)
      |   |------
⋮     |   | ⋮
n     |   | C
n + 1 | C          ∃E, p, m–n
```


Strategy for building a derivation

1. Check each premise and conclusion: see what rules are suggested by each WFF. Premises suggest E-rules; conclusions suggest I-rules. Use the main connective in a WFF to determine which E or I rule.
2. Some rules may be used immediately, some require extra data. Do the immediate ones immediately, and for the others, if the extra data is not available yet, take note of that fact and use the rule once it becomes available. (See the table below.)
3. Once you've used what rules you can (from the previous step), go back to the beginning and check your new premises and conclusions. Go on with this process till you are finished.
4. Remember anytime you get $\perp$ (and if that's not the conclusion you are aiming at), you can write any desired conclusion after that using the ($\perp E$) rule.
5. If there is no rule you can use, and you are not finished, try replacing the conclusion C with $\neg\neg C$, and use the ($\neg I$) rule to derive $\neg\neg C$ (and of course the ($\neg\neg E$) rule to get to your real conclusion C).

Immediate and not immediate rules:

Immediate – do indicated action	**Not immediate** – look & wait for required data
($\wedge E$) Write one or both conjuncts	($\wedge I$) Need both conjuncts
($\longrightarrow I$) Write subderivation for $p \longrightarrow q$: p at top, q at bottom	($\longrightarrow E$) Need p as well as $p \longrightarrow q$
($\neg I$) Write subderivation for $\neg p$: p at top, $\perp$ at bottom	($\neg E$) Need p as well as $\neg p$
($\vee E$) Write conclusion and two subderivations for $p \vee q$: p at top of one, q at top of other, conclusion at bottom	($\vee I$) Need one or the other disjunct
($\forall I$) Write subderivation with new "fresh" variable u (no new premise; conclusion $P(u)$)	($\forall E$) Look for a suitable name t for the bound variable
($\exists E$) Write subderivation with new "fresh" variable u (with new premise $P(u)$; conclusion as in derivation)	($\exists I$) Need $P(t)$ with a suitable name t for the bound variable