SAY, BUD, WHERE DO YOU THINK YOU ARE GOING?

We'll number the buds 0, 1, 2, . . . , in order of their appearance. In the plan view of Figure 4.31(b), we've put bud number 0 at the 12 o'clock position, and since it was born first, it has already arrived at the perimeter. Buds 0 and 1 separate the cone into larger and smaller sectors. Bud 2 finds it easier to exist in the larger sector, forcing 3 into the smaller one. Whereabouts in these sectors will they be? Since number 1 is newer and nearer the tip than number 0, it is likely to be more inhibiting, so numbers 2 and 3 will be slightly nearer to 0 than they are to 1. We've put them at about 9 o'clock and 2 o'clock in Figures 4.31(b) and (c). At this stage we have four sectors, the largest of which is between buds 1 and 2. We expect number 4 to be born in this sector, slightly nearer to 1, since 2 is more recent.

This process might seem rather rough, but it is in fact extremely stable. Suppose, for instance, that bud number 1 was born more or less exactly opposite number 0, leaving two equal sectors open for number 2. Number 2 would randomly choose one of these, and, as it grew, it would push away from 0 and 1 (but slightly more from 1) and so enlarge its own sector. Similar phenomena happen at later stages in the process. The reader should also understand that the action takes place at a very early stage in the development, so that it is likely to be in an otherwise uniform environment, maybe only a few millimeters across.

Figure 4.32 shows what happens when many buds have developed. This time we've taken an idealized plant in which the cone is nearly cylindrical, and we've unrolled the cylinder. Of course, a real plant won't have the buds in exactly the "right" places, but neither can they be too far off. The dotted ellipses in Figure 4.32 show how the spheres of influence of buds 17, 20, and 22 leave a ready-made hole for bud number 25. It won't be born too far from the center of this hole, and even if it's a little out of position, as it grows it will jostle itself into the right position. Once the process is established, it's really very hard for it to go wrong.

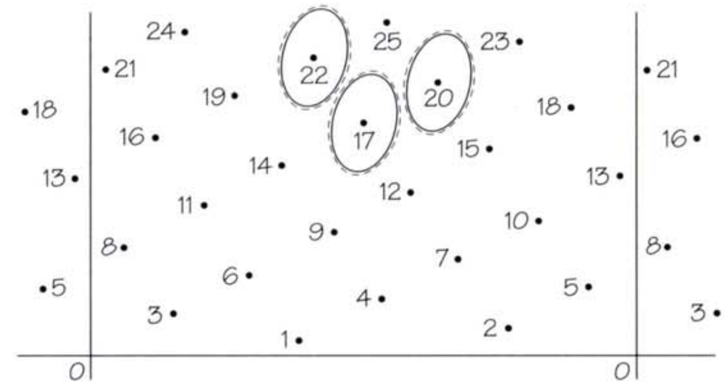

FIGURE 4.32 *The first twenty-five buds.*

In the perfect version of the process, each new bud advances by the same angle, the angle being that which divides a complete turn in a golden section. In our figures this is 0.618 of a counterclockwise revolution, or 0.382 of a clockwise one.

How many spirals are there? These spirals are very much in the eye of the beholder. You tend to link up neighboring buds into a chain. Thus, in Figure 4.32, you probably mentally link the buds into lines with difference 3 that slope up to the right, and lines of difference 5 that slant up to the left.

On the other hand, if you squint up the page from the bottom, you may find it easier to organize them into lines of difference 8, or even of 13. The successive Fibonacci numbers arise as those multiples of the basic angle that are nearest to whole numbers of revolutions. The particular Fibonacci numbers you notice depend on how squashed the vertical scale is, compared with the horizontal. Figures 4.33 and 4.34 are just Figures 4.32 with the vertical axis more and more squashed, and cut into domains by assigning each point to its nearest bud.

Although it looks very much like a plant, Figure 4.35 was actually produced just by applying this rule from a simple mathematical formula that squashes the vertical axis by a gradually varying amount.

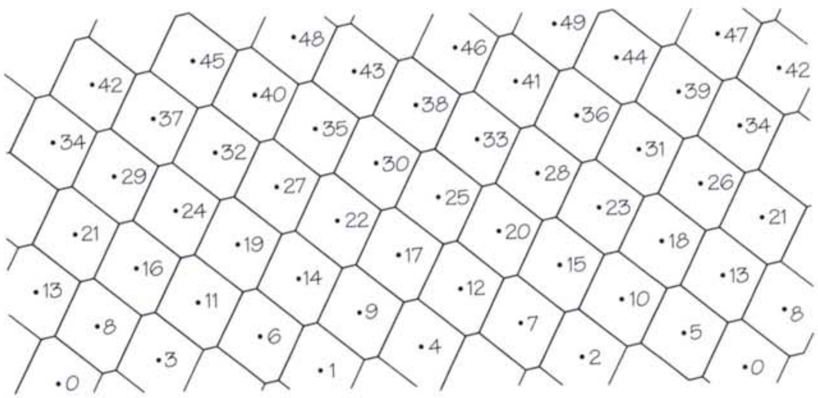

FIGURE 4.33 *Here region n touches n ± 5, n ± 8, and n ± 13.*

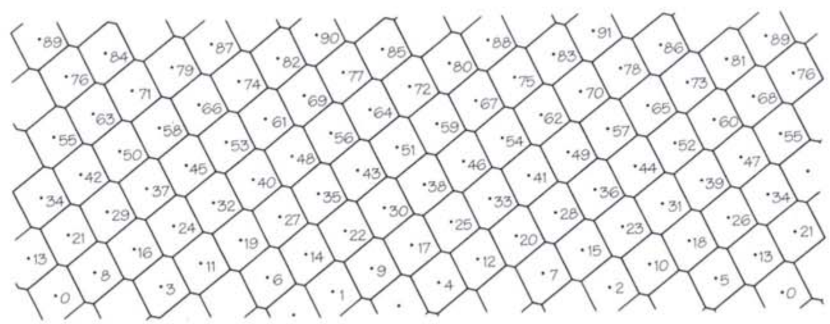

FIGURE 4.34 *Here n touches n ± 8, n ± 13, and n ± 21.*

In Figure 4.32, the numbers 3, 5, and 8 are most obvious; in Figure 4.33, with the scale multiplied by a half, 5 and 8 are most noticeable; in Figure 4.34, with scale only one-fifth, 8, 13, (and 21) are most obvious. At the foot of Figure 4.35, the numbers 3, 5, and 8 are prominent; at the top, 13 and 21 begin to predominate.

Before you next eat a pineapple, try to find the correct numbering

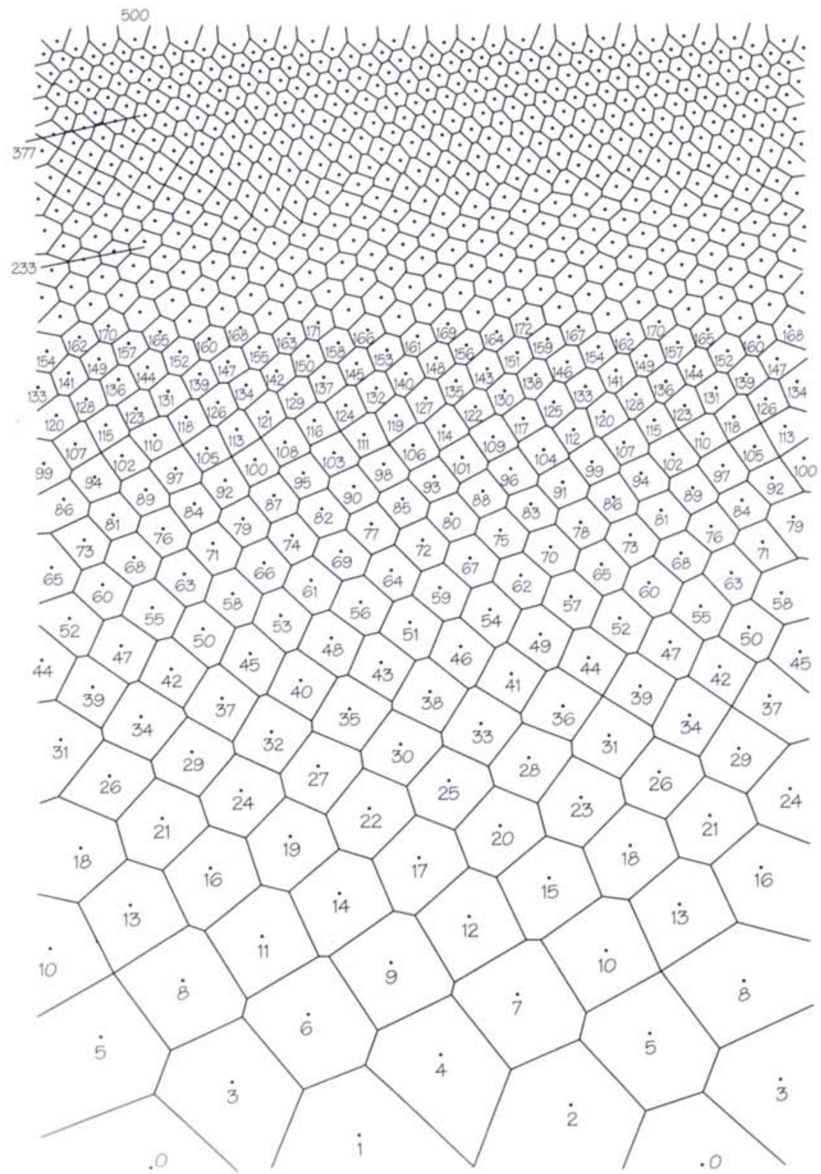

FIGURE 4.35 *Variable growth rate accentuates different Fibonacci numbers. (Diagram courtesy of Len Bos.)*

of the buds on it. The pattern is usually easiest to follow about halfway up, but with a little care you can work backward to the base and even identify number 0.

The leaves on the stems of plants exhibit remnants of this process. The embryo leaf buds in such cases were originally stacked quite tightly around the stem, as in our figures. In a later phase of growth the stem elongates so that all that is left of the original arrangement is a tendency for each leaf to spiral about 0.382 (or roughly 2/5) of a revolution from its predecessor.

The petals of a rose exhibit the same phenomenon in the way they overlap their predecessors, but they are usually so tightly packed that the structure is hard to see without dissecting it. Figure 4.36 shows a flower on which we've numbered the petals individually.

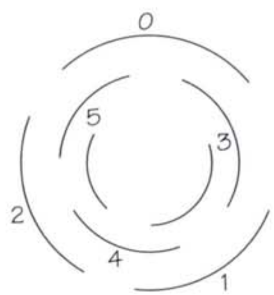

FIGURE 4.36 *The arrangement of the petals of a flower.*

We may even speculate that the exact number (when it is small) of petals on a flower might be determined by such a mechanism. The following table shows in fractions of a revolution the angle defining the position of petal number n (this is just the fractional part of $n\tau$) and also the smallest angle between this petal and any of its predecessors. If the latter were the only relevant parameter, it would estimate the likelihood of the emergence of petal number n.

Petal number	0	1	2	3	4	5	6	7	8	
Position angle	0	.618	.236	.854	.472	.090	.708	.326	.944	...
Smallest angle		.382	.236	.146	.146	.090	.090	.090	.056	...

The position angles are just the fractional parts of successive multiples

$$0,\ 1.618\ldots,\ 3.236\ldots,\ 4.854\ldots$$

of the **golden number**

$$\tau = \frac{1+\sqrt{5}}{2} = 1.6180339887494\ldots .$$

On this view you will see that the fifth petal (number 4) finds things just as easily (0.146) as its predecessor, while the sixth and later ones have a substantially harder time (0.090). Without being able to count, but merely by setting a given level of inhibition, a flower can arrange for the total number of petals to be a small Fibonacci number, but not any other number!

It is not so easy to control things in order to get a larger Fibonacci number exactly, since the influence of petal number 0 becomes rapidly less noticeable. In fact, when a flower has a large number of petals, the exact number of petals usually depends on the particular specimen.

The reader should beware of trying to force this theory too far, because plants use many other mechanisms in their development. Some emit pairs of leaves simultaneously on opposite sides of the stem. Successive pairs may then rotate by 0.191 (half of 0.382) of a revolution, but they are just as likely to rotate by a right angle. Many flowers have exactly six petals. For these it is usually the case that the petals are organized as two generations of three petals each. This effect shows very clearly on a narcissus; actually, the first three of its "petals" are *sepals* rather than petals.

One occasionally sees sport pinecones in which the numbers of spirals are the doubles of Fibonacci numbers. Presumably this happened because buds 0 and 1 emerge almost simultaneously on opposite sides and the subsequent process was bisected.

The next time you eat a head of cauliflower, notice that not only does it have Fibonacci numbers of spirals of florets, but these in turn often have spirals of subflorets. And on your walk through the forest,

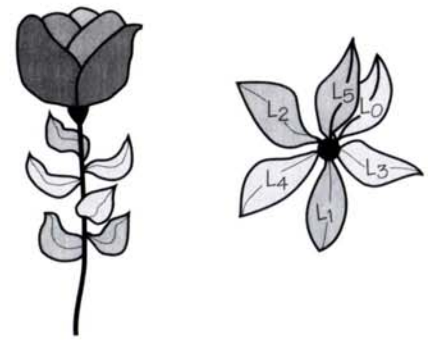

FIGURE 4.37 *Leaves round a stem.*

look down the stem of a plant. If two leaves appear to be above one another (Figure 4.37), they are probably a Fibonacci number apart.

The arrangement of the larger branches on a tree is often more random, but there are some species of mountain trees in which the Fibonacci organization can be seen throughout the entire structure, even in the roots, if you dig down to them.

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