If there are f_n pairs of rabbits in the nth generation, then

$$f_1 = 1 \text{ (the original pair)},$$

$$f_2 = 1 \text{ (their immediate progeny)},$$

$$f_{n+2} = f_n + f_{n+1},$$

since we get a pair in generation $n + 2$ for each pair in generation n or generation $n + 1$ (see Figures 4.22 and 4.23).

$$f_0 = 0\ 1\ 1\ 2\ 3\ 5\ 8\ 13\ 21\ 34\ 55\ 89\ 144\ 233\ 377\ 610\ \ldots$$

are called **Fibonacci numbers**, since Leonardo's father was nicknamed Bonacci, and so Leonardo was Fibonacci (*filius Bonacci* = "son of the good-natured one"). Fibonacci numbers arise in so many ways, it's almost unbelievable: their manifestations seem as numerous as Leonardo's rabbits. There is even a mathematical periodical, the *Fibonacci Quarterly*, devoted entirely to the subject. We'll only mention a few of the more striking properties.

The **Lucas numbers**, l_n,

$$l_0 = 2\ 1\ 3\ 4\ 7\ 11\ 18\ 29\ 47\ 76\ 123\ 199\ 322\ 521\ 843\ 1364\ \ldots$$

(defined by the same rule, but with a different start) are related to the Fibonacci numbers in many ways.

$$\begin{aligned} f_{2n} &= f_n l_n & l_{2n} &= l_n^2 - 2(-1)^n \\ f_0 + f_1 + \cdots + f_n &= f_{n+2} - 1 & l_0 + l_1 + \cdots + l_n &= l_{n+2} - 1 \\ l_n &= f_{n-1} + f_{n+1} & 5f_n &= l_{n-1} + l_{n+1} \\ 2f_{m+n} &= f_m l_n + f_n l_m & 2l_{m+n} &= l_m l_n + 5f_m f_n \end{aligned}$$

$$f_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

This last relation, noticed by Lucas, shows that you can read the Fibonacci numbers from Pascal's triangle (Figure 4.24).

Kepler pointed out that the ratios of consecutive Fibonacci numbers approach 1.618... . The exact limit is the **golden number**,

$$\tau = \frac{1 + \sqrt{5}}{2} = 1.6180339887498948482\ldots$$

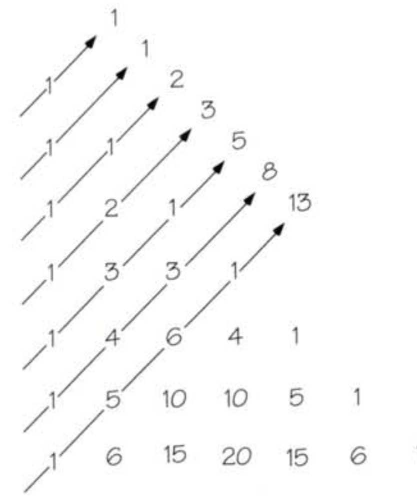


FIGURE **4.24** *Summing diagonals in Pascal's triangle gives Fibonacci numbers.*

which was known to the Greeks. The ratios of consecutive Lucas numbers tend to the same limit, while the ratio, l_n/f_n, between corresponding Lucas and Fibonacci numbers, tends to $\sqrt{5}$. In fact, there are formulas for f_n and l_n in terms of the golden number, which we'll give in Chapter 7.

Leonardo's rabbit problem was, of course, not too true to life. However, the Fibonacci numbers really do occur in nature! You can read them off pineapples or cacti, off pinecones and sunflowers: they also control the arrangements of leaves of almost all plants.

PHYLLOTAXIS

The botanical name for leaf arrangement is *phyllotaxis*. Look at the florets in the sunflower's head in Figure 4.25. They appear to form two systems of spirals, radiating from the center. Although it looks symmetrical, the numbers of clockwise and counterclockwise spirals are in fact not equal. If you count them carefully, you'll find 55 clockwise spirals and 34 counterclockwise ones.

FIGURE 4.25 *Sunflower. (Courtesy of D.R. Fowler and P. Prusinkiewicz.)*

The pineapples in Figure 4.26 and the daisy in Figure 4.27 exhibit a similar phenomenon. In Figure 4.28 which is a tracing of Figure 4.27, we emphasize the 21 such spirals going one way and 34 going the other way by numbering the petals. You can see such spirals on many other plants, such as cauliflowers (Figure 4.29), pinecones and certain kinds of cactus. There are usually two systems of florets, seeds, twigs, petals, or whatever,[1] going in opposite directions, and the numbers of spirals in these systems are consecutive Fibonacci numbers. Why is this so?

We'll describe what happens at an earlier stage in the plant's life. We regard the tip of a plant as a cone and consider the initial arrival of "buds" on this cone. The cone may be very flat, as in the sunflower (Figure 4.30(a)), or pointed, as in the stem (Figure 4.30(b)), or somewhere in between, as in the pineapple of Figure 4.30(c).

[1] "Everything is a leaf" —Goethe.

FIGURE 4.26 *Spirals on pineapples. (Photo courtesy of D.R. Fowler and A. Snider.)*

FIGURE 4.27 *Daisy capitulum. (Computer generated by Deborah R. Fowler on an IRIS workstation using an L-grammar algorithm.)*

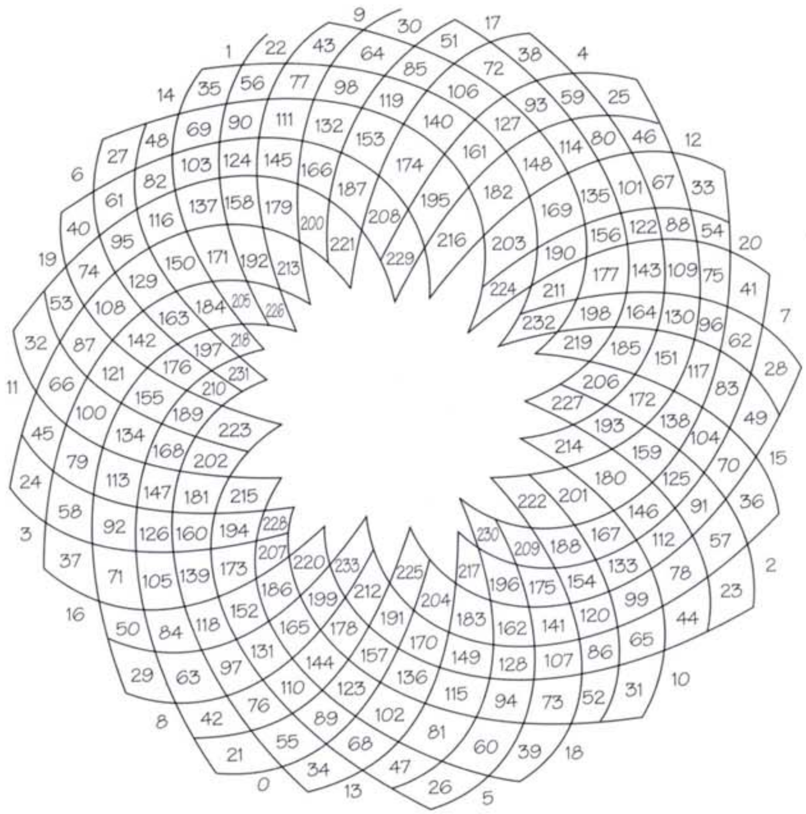


FIGURE 4.28 *The florets of Figure 4.25 numbered to show arithmetic progressions with Fibonacci differences 21 and 34.*

These buds may become seeds in a seed case, twigs on a branch, petals of a flower, and so forth. However, at this stage of its life, one bud inhibits or repels others. This may be because they're crowded and physically push each other apart, or maybe they are competing for essential nutrients, or perhaps they are deliberately secreting some inhibiting substance.

We regard the growth of our plant as taking place at the tip of the cone. Since the tip is continually advancing by new growth, a given portion of the plant moves steadily downward and outward relative to the tip (Figure 4.31(a)).

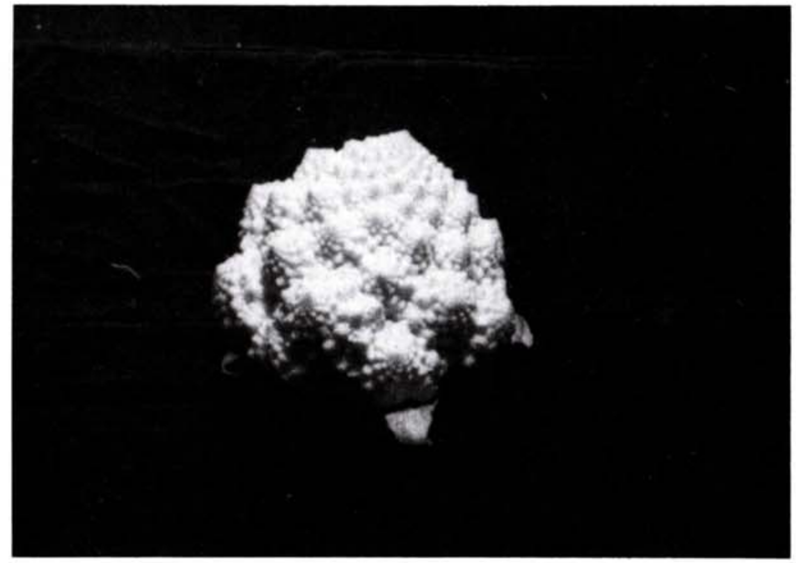
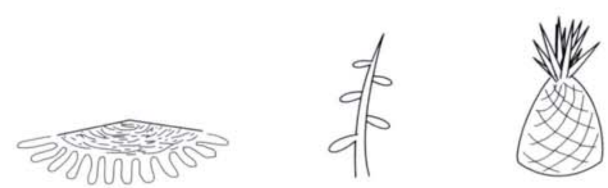
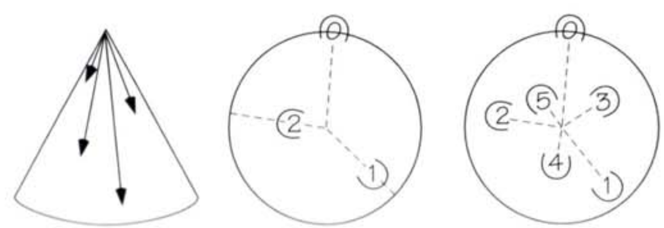


FIGURE 4.29 *The inflorescence of a cauliflower. (Photo courtesy of E. Thirian.)*



(a) (b) (c)

FIGURE 4.30 *The "cones" of various plants.*



(a) (b) (c)

FIGURE 4.31 *Buds distributing themselves on a conical plant*