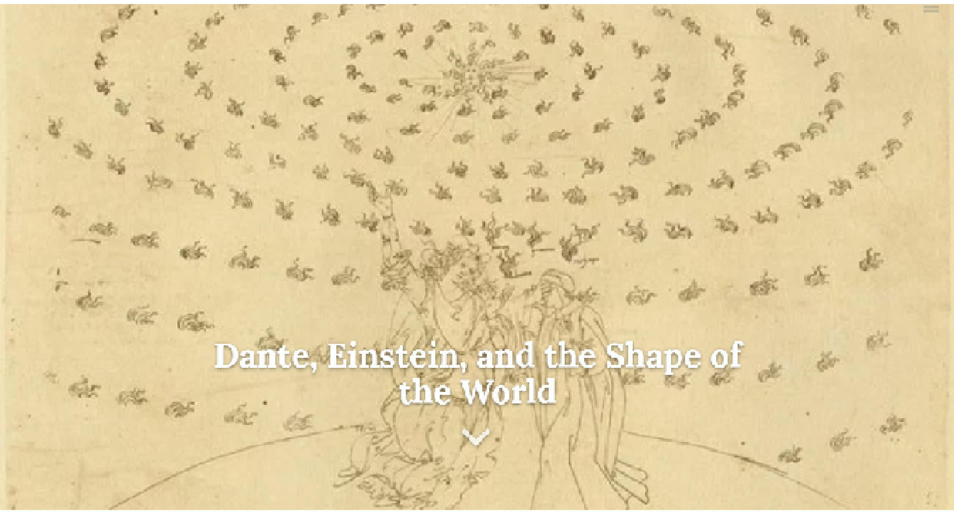


Last week, we began a series of posts dedicated to thinking about immortality. If we want to even pretend to think precisely about immortality, we will have to consider some fundamental questions. What does it mean to be immortal? What does it mean to live forever? Are these the same thing? And since immortality is inextricably tied up in one's relationship with time, we must think about the nature of time itself. Is there a difference between external time and personal time? What is the shape of time? Is time linear? Circular? Finite? Infinite?

Of course, we exist not just across time but across space as well, so the same questions become relevant when asked about space. What is the shape of space? Is it finite? Infinite? It is not hard to see how this question would have a significant bearing on our thinking about immortality. In a finite universe (or, more precisely, a universe in which only finitely many different configurations of matter are possible), an immortal being would encounter the same situations over and over again, would think the same thoughts over and over again, would have the same conversations over and over again. Would such a life be desirable? (It is not clear that this repetition would be avoidable even in an infinite universe, but more on that later.)

Today, we are going to take a little historical detour to look at the shape of the universe, a trip that will take us from Ptolemy to Dante to Einstein, a trip that will uncover a remarkable confluence of poetry and physics.

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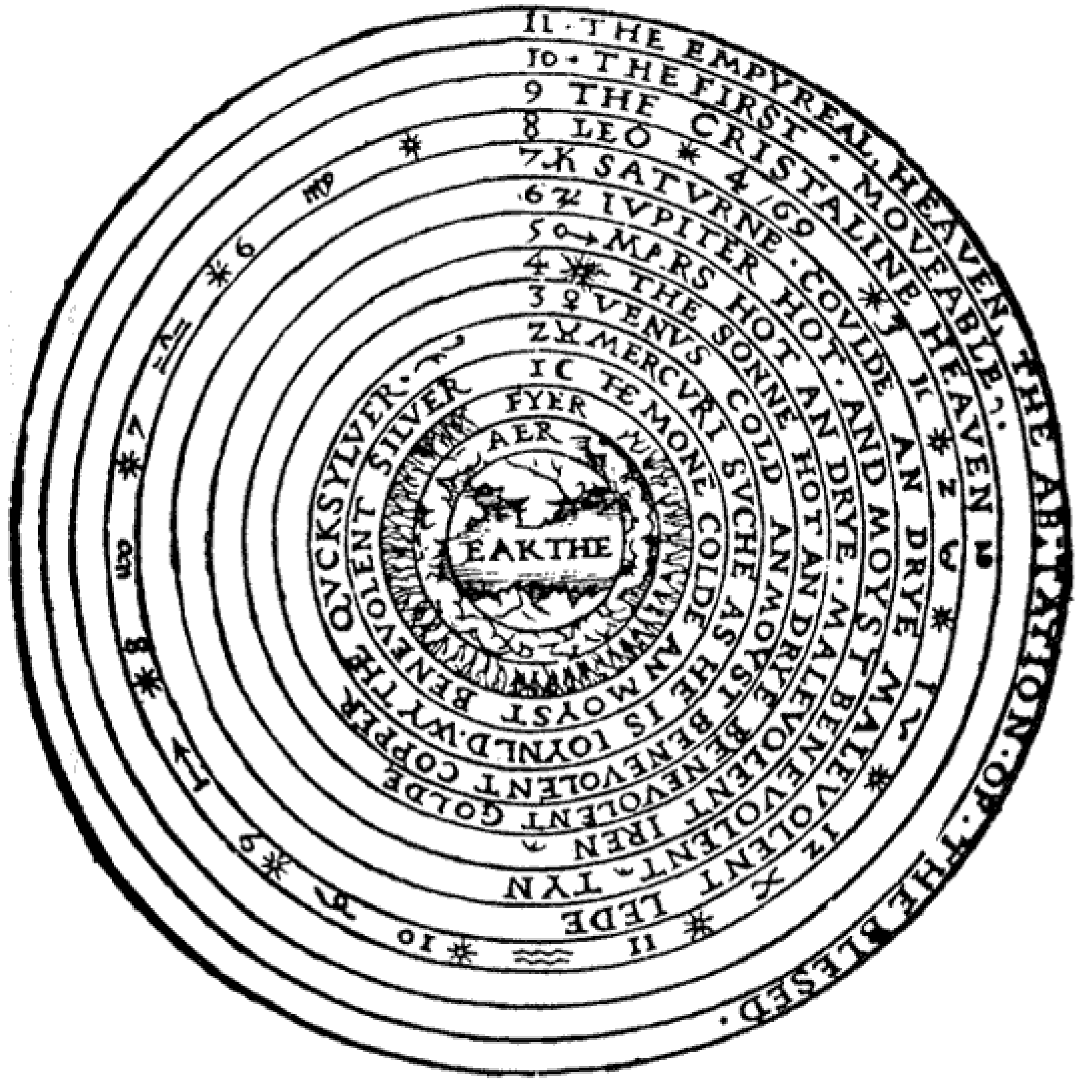
One of the dominant cosmological views from ancient Greece and the Middle Ages was that of the Ptolemaic, or Aristotelian, universe. In this image of the world, Earth is the fixed, immobile center of the universe, surrounded by concentric, rotating spheres. The first seven of these spheres contain the seven "planets": the Moon, Mercury, Venus, the Sun, Mars, Jupiter, and Saturn. Surrounding these spheres is a sphere containing the fixed stars. This is the outermost sphere visible from Earth, but there is still another sphere outside it: the Primum Mobile, or "Prime Mover," which gives motion to all of the spheres inside it. (In some accounts the Primum Mobile is itself divided into three concentric spheres: the Crystalline Heaven, the First Moveable, and the Empyrean. In some other accounts, the Empyrean (higher heaven, which, in the Christianity of the Middle Ages, became the realm of God and the angels) exists outside of the Primum Mobile.)




*An illustration of the Ptolemaic universe from The Fyrst Boke of the Introduction of Knowledge by Andrew Boorde (1542)*

This account is naturally vulnerable to an obvious question, a question which, though not exactly in the context of Ptolemaic cosmology, occupied me as a child lying awake at night and was famously asked by Archytas of Tarentum, a Greek philosopher from the fifth century BC: If the universe has an edge (the edge of the outermost sphere, in the Ptolemaic account), then what lies beyond that edge? One could of course assert that the Empyrean exists as an infinite space outside of the Primum Mobile, but this would run into two objections in the intellectual climate of both ancient Greece and Europe of the Middle Ages: it would compromise the aesthetically pleasing geometric image of the universe as a finite sequence of nested spheres, and it would go against a strong antipathy towards the infinite. Archytas' question went largely unaddressed for almost two millennia, until Dante Alighieri, in the *Divine Comedy*, proposed a novel and prescient solution.

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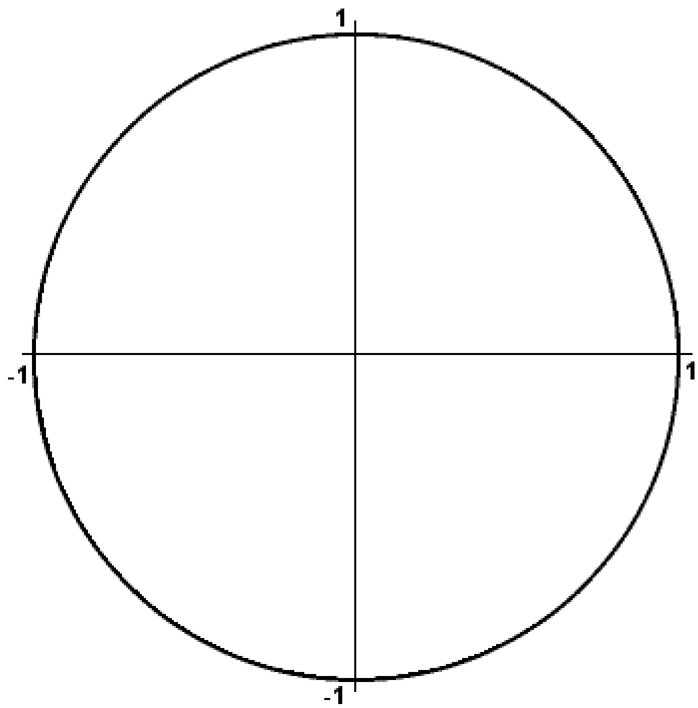
Before we dig into Dante, a quick mathematical lesson on generalized spheres. For a natural number $n$, an $n$-sphere is an $n$-dimensional manifold (i.e. a space which, at every point, locally looks like $n$-dimensional real Euclidean space) that is most easily represented, embedded in $n+1$-dimensional space, as the set of all points at some fixed positive distance (the "radius" of the sphere) from a given "center point."

Perhaps some examples will clarify this definition. Let us consider, for various values of $n$, the $n$-sphere defined as the set of points in $(n+1)$-dimensional Euclidean space at distance 1 from the origin (i.e. the point (0,0,…,0)).

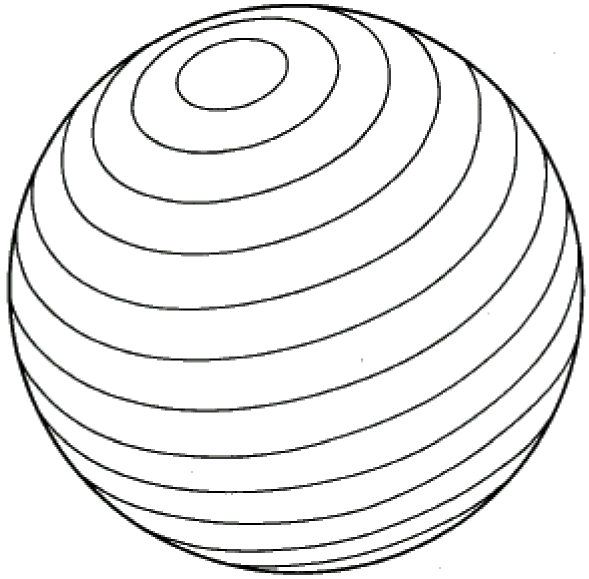
If $n=0$, this is the set of real numbers whose distance from 0 is equal to 1, which is simply two points: 1 and -1.

If $n=1$, this is just the set of points $(x, y)$ in the plane at a distance of 1 from $(0, 0)$. This is the circle, centered at the origin, with radius 1.




*A 1-sphere*

If $n=2$, this is the set of points $(x, y, z)$ in 3-dimensional space at a distance 1 from the point $(0, 0, 0)$. This is the surface of a ball of radius 1, and is precisely the space typically conjured by the word "sphere."



*A 2-sphere*

0-, 1-, and 2-spheres are all familiar objects; beyond this, we lose some ability to visualize $n$-spheres due to the difficulty of considering more than three spatial dimensions, but there are useful ways to think about higher-dimensional spheres by analogy with the more tangible lower-dimensional ones. Let us try to use these ideas to get some understanding of the 3-sphere.

First, note that, for a natural number $n$, the non-trivial "cross-sections" of an $n + 1$-sphere are themselves $n$-spheres! For example, if a 1-sphere (i.e. circle) is intersected with a 1-dimensional Euclidean space (a line) in a non-trivial way, the result is a $0$-sphere (i.e. a pair of points). If a 2-sphere is intersected with a 2-dimensional Euclidean space (a plane) in a non-trivial way, the result is a 1-sphere (this is illustrated above in our picture of a 2-sphere). The same relationship holds for higher dimensional spheres: if a 3-sphere is intersected with a 3-dimensional Euclidean space in a non-trivial way, the result is a 2-sphere.

Suppose that you are a 2-dimensional person living in a 2-sphere universe. Let's suppose, in fact, that you are living in the 2-sphere pictured above, with the 1-sphere "latitude lines" helpfully marked out for you. Let's suppose that you begin at the "north pole" (i.e. the point at the top, in the center of the highest circle) and start moving in a fixed direction. At fixed intervals, you will encounter the 1-sphere latitude lines. For a while, these 1-spheres will be increasing in radius. This will make intuitive sense to you. You are moving "further out" in space; each successive circle "contains" the last and thus should be larger in radius. After you pass the "equator," though, something curious starts happening. Even though you haven't changed direction and still seem to be moving "further out," the radii of the circles you encounter start shrinking. Eventually, you reach the "south pole." You continue on your trip. The circles wax and wane in a now familiar way, and, finally, you return to where you started.

A similar story could be told about a 3-dimensional being exploring a 3-sphere. In fact, I think we could imagine this somewhat easily. Suppose that we in fact live in a 3-sphere. For illustration, let us place a "pole" of this 3-sphere at the center of the Earth. Now suppose that we, in some sort of tunnel-boring spaceship, begin at the center of the Earth and start moving in a fixed direction. For a while, we will encounter 2-sphere cross-sections of increasing radius. Of course, in the real world these are not explicitly marked (although, for a while, they can be nicely represented by the spherical layers of the Earth's core and mantle, then the Earth's surface, then the sphere marking the edge of the Earth's atmosphere) but suppose that, in our imaginary world, someone has helpfully marked them. For a while, these successive 2-spheres have larger and larger radii, as is natural. Eventually, of course, they will start to shrink, contracting to a point before expanding and contracting as we return to our starting point at the Earth's core.

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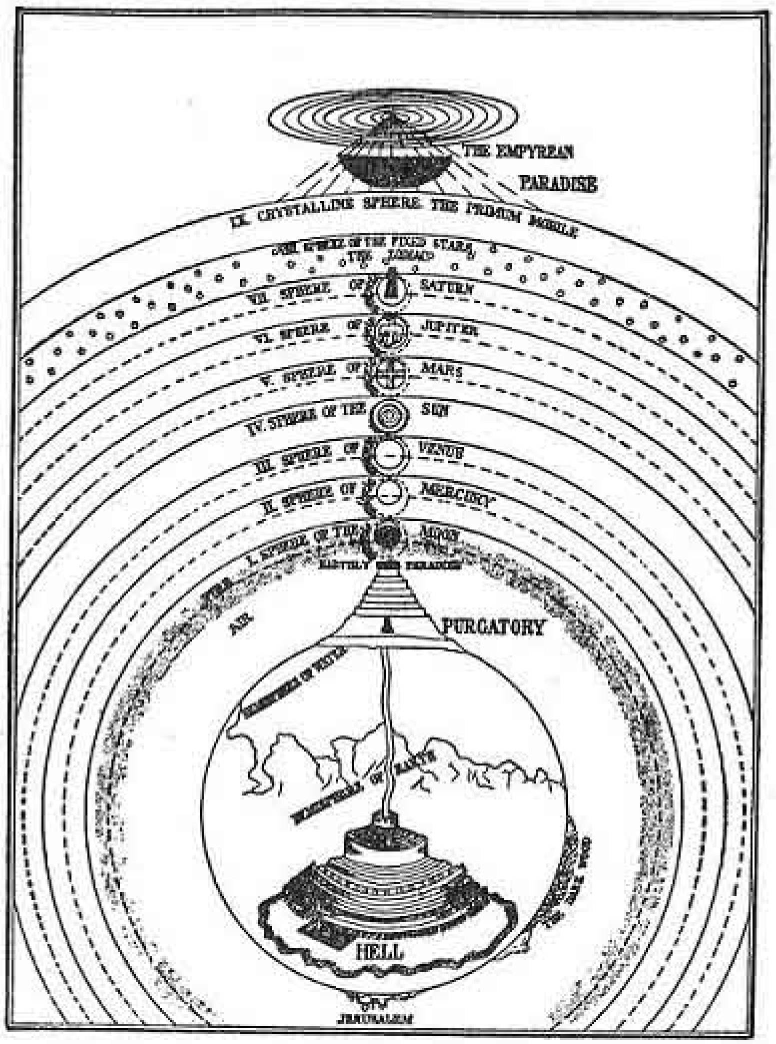
Dante's *Divine Comedy*, completed in 1320, is one of the great works of literature. In the first volume, *Inferno*, Dante is guided by Virgil through Hell, which exists inside the Earth, directly below Jerusalem (from where I happen to be writing this post). In the second volume, *Purgatorio,* Virgil leads Dante up Mount Purgatory, which is situated antipodally to Jerusalem and formed of the earth displaced by the creation of Hell. In the third volume, *Paradiso*, Dante swaps out Virgil for Beatrice and ascends from the peak of Mount Purgatory towards the heavens.




*Dante's universe. Image by Michelangelo Caetani.*

Dante's conception of the universe is largely Ptolemaic, and most of *Paradiso* is spent traveling outward through the larger and larger spheres encircling the Earth. In Canto 28, Dante reaches the Primum Mobile and turns his attention outward to what lies beyond it. We are finally in a position to receive an answer to Archytas' question, and the answer that Dante comes up with is surprising and elegant.

The structure of the Empyrean, which lies outside the Primum Mobile, is in large part a mirror image of the structure of the Ptolemaic universe, a revelation that is foreshadowed in the opening stanzas of the canto:

When she who makes my mind imparadised
Had told me of the truth that goes against
The present life of miserable mortals —

As someone who can notice in a mirror
A candle's flame when it is lit behind him
Before he has a sight or thought of it,

And turns around to see if what the mirror
Tells him is true, and sees that it agrees
With it as notes are sung to music's measure —

Even so I acted, as I well remember,
While gazing into the bright eyes of beauty
With which Love wove the cord to capture me.

When Dante looks into the Empyrean, he sees a sequence of concentric spheres, centered around an impossibly bright and dense point of light, expanding to meet him at the edge of the Primum Mobile:

I saw a Point that radiated light
So sharply that the eyelids which it flares on
Must close because of its intensity.

Whatever star looks smallest from the earth
Would look more like a moon if placed beside it,
As star is set next to another star.

Perhaps as close a halo seems to circle
The starlight radiance that paints it there
Around the thickest mists surrounding it,

As close a ring of fire spun about
The Point so fast that it would have outstripped
The motion orbiting the world most swiftly.

And this sphere was encircled by another,
That by a third, and the third by a fourth,
The fourth by a fifth, the fifth then by a sixth.

The seventh followed, by now spread so wide
That the whole arc of Juno's messenger
Would be too narrow to encompass it.

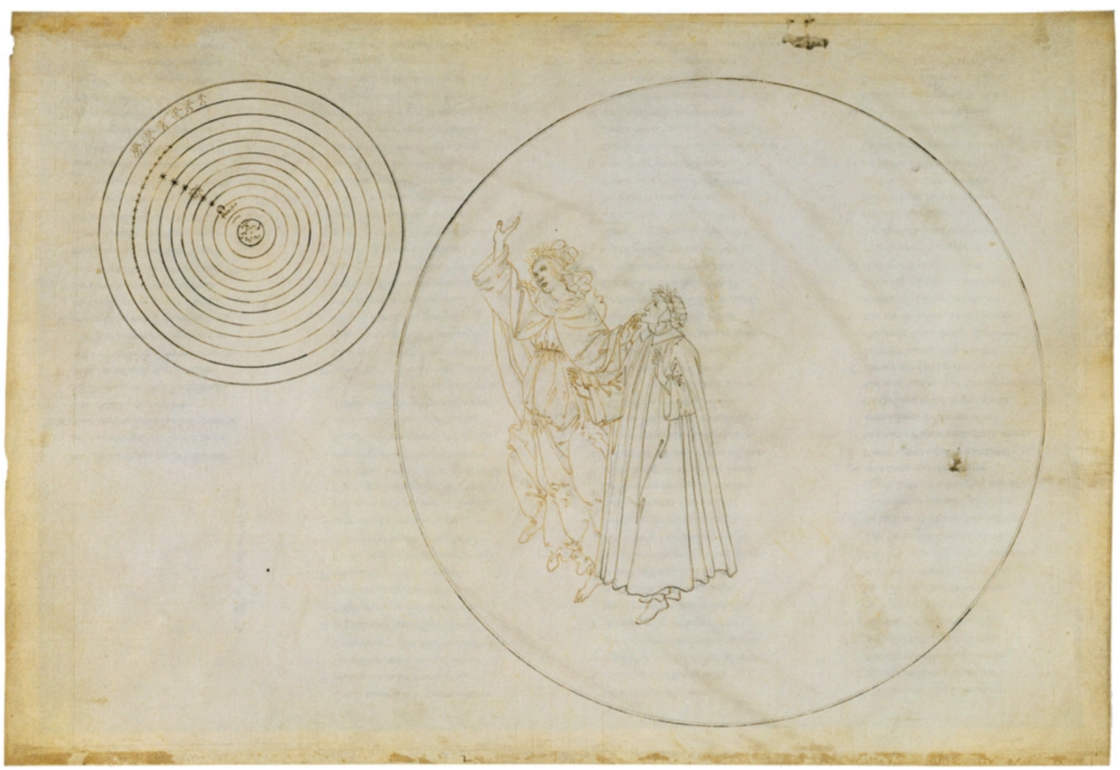
So too the eighth and ninth, and each of them
Revolved more slowly in proportion to
The number of turns distant from the center.

This seemingly obscure final detail, that the spheres of the Empyrean spin increasingly slowly as they increase in size, and in distance from the point of light, turns out to be important. Dante is initially confused because, in the part of the Ptolemaic universe from the Earth out to the Primum Mobile, the spheres spin faster the larger they are; the fact that this is different in the Empyrean seems to break the nice symmetry he observes. Beatrice has a ready explanation, though: the overarching rule governing the speed at which the heavenly spheres rotate is not based on their size, but rather on their distance from God.

This is a telling explanation and seems to confirm that the picture Dante is painting of the universe is precisely that of a 3-sphere, with Satan, at the center of the Earth, at one pole and God, in the point of light, at the other. If Dante continues his outward journey from the edge of the Primum Mobile, he will pass through the spheres of the Empyrean in order of decreasing size, arriving finally at God. Note that this matches precisely the description given above of what it would be like to travel in a 3-sphere. Dante even helpfully provides a fourth dimension into which his 3-sphere universe is embedded: not a spatial dimension, but a dimension corresponding to speed of rotation!

(For completeness, let me mention that the spheres of the Empyrean are, in order of decreasing size and hence increasing proximity to God: Angels, Archangels, Principalities, Powers, Virtues, Dominions, Thrones, Cherubim, and Seraphim.)

Dante's ingenious description of a finite universe helped the Church to argue against the existence of the infinite in the physical world. Throughout the Renaissance, Scientific Revolution, and Enlightenment, this position was gradually eroded in favor an increasingly accepted picture of infinite, flat space. A new surprise awaited, though, in the twentieth century.



*Beatrice explaining the nature of the heavens to Dante. Drawing by Botticelli.*

In 1917, Einstein revolutionized cosmology with the introduction of general relativity, which provided an explanation of gravity as arising from geometric properties of space and time. Central to the theory are what are now known as the Einstein Field Equations, a system of equations that describes how gravity interacts with the curvature of space and time caused by the presence of mass and energy. In the 1920s, an exact solution to the field equations, under the assumptions that the universe is *homogeneous* and *isotropic* (roughly, has laws that are independent of absolute position and orientation, respectively), was isolated. This solution is known as the Friedmann-Lemaître-Robertson-Walker metric, after the four scientists who (independently) derived and analyzed the solution, and is given by the equation,

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right),$$

where $k$ is a constant corresponding to the "curvature" of the universe. If $k = 0$, then the FLRW metric describes an infinite, "flat" Euclidean universe. If $k < 0$, then the metric describes an infinite, hyperbolic universe. If $k > 0$, though, the metric describes a finite universe: a 3-sphere.

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PS: Andrew Boorde, from whose book the above illustration of the Ptolemaic universe is taken, is a fascinating character. A young member of the Carthusian order, he was absolved from his vows in 1529, at the age of 39, as he was unable to adhere to the "rugorosite" of religion. He turned to medicine, and, in 1536, was sent by Thomas Cromwell on an expedition to determine foreign sentiment towards King Henry VIII. His travels took him throughout Europe and, eventually, to Jerusalem, and led to the writing of the *Fyrst Boke of the Introduction of Knowledge,* perhaps the earliest European guidebook. Also attributed to him (likely without merit) is *Scoggin's Jests, Full of Witty Mirth and Pleasant Shifts, Done by him in France and Other Places, Being a Preservative against Melancholy*, a book which, along with Boord himself, plays a key role in Nicola Barker's excellent novel, Darkmans.

Further Reading:

Mark A. Peterson, "Dante and the 3-sphere," American Journal of Physics, 1979.

Carlo Rovelli, "Some Considerations on Infinity in Physics," and Anthony Aguirre, "Cosmological Intimations of Infinity," both in *Infinity: New Research Frontiers,* edited by Michael Heller and W. Hugh Woodin.

Cover Image: Botticelli's drawing of the Fixed Stars.

**3-sphere** **aristotle** **botticelli** **cosmology** **dante** **divine comedy** **einstein** **finite** **general relativity** **infinite** **infinity** **literature** **physics** **poetry** **ptolemy** **universe**

Posted on February 23, 2017 by Chris Lambie-Hanson in Art, History, literature, Mathematics, Philosophy, Physics

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