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Maths & Logic (360-124)
More quantifier exercises - Solutions

(Q1)

1	$\forall x \forall y P(x,y)$	
2	$\forall y P(\mathsf{a}, y)$	$(\forall\mathrm{E})$, 1
3	$P(\mathsf{a},\mathsf{a})$	$(\forall\mathrm{E})$, 2

(Q2)

1	$\forall x \forall y P(x,y)$	
2	$u \quad \vert\ \forall y P(u,y)$	$(\forall\mathrm{E})$, 1
3	$\quad \vert\ P(u,u)$	$(\forall\mathrm{E})$, 2
4	$\forall x P(x,x)$	$(\forall\mathrm{I})$, 2–3

(Q3)

1	$\forall x (P(x) \wedge Q(x) \to R(x))$	
2	$Q(\mathsf{a}) \wedge \forall z P(z)$	
3	$Q(\mathsf{a})$	$(\wedge\mathrm{E})$, 2
4	$\forall z P(z)$	$(\wedge\mathrm{E})$, 2
5	$P(\mathsf{a})$	$(\forall\mathrm{E})$, 4
6	$P(\mathsf{a}) \wedge Q(\mathsf{a})$	$(\wedge\mathrm{I})$, 3, 5
7	$P(\mathsf{a}) \wedge Q(\mathsf{a}) \to R(\mathsf{a})$	$(\forall\mathrm{E})$, 1
8	$R(\mathsf{a})$	$(\to\mathrm{E})$, 6, 7
9	$P(\mathsf{a}) \wedge R(\mathsf{a})$	$(\wedge\mathrm{I})$, 5, 8

(Q4)

1	$\exists x P(x)$	
2	$\quad \vert\ \forall x Q(x)$	
3	$\quad \vert\ u \quad \vert\ P(u)$	
4	$\quad \vert \quad \vert\ Q(u)$	$(\forall\mathrm{E})$, 2
5	$\quad \vert \quad \vert\ P(u) \wedge Q(u)$	$(\wedge\mathrm{I})$, 3, 4
6	$\quad \vert \quad \vert\ \exists x (P(x) \wedge Q(x))$	$(\exists\mathrm{I})$, 5
7	$\quad \vert\ \exists x (P(x) \wedge Q(x))$	$(\exists\mathrm{E})$, 1, 3–6
8	$\forall x Q(x) \to \exists x (P(x) \wedge Q(x))$	$(\to\mathrm{I})$, 2–7

Notice in Q5, in lines 7,8 the $(\forall\mathrm{E})$ rule is used twice each.

(Q5)

1	$\forall x \forall y (R(x,y) \to (P(x) \wedge \neg P(y)))$	
2	$\exists x \exists y (R(x,y) \wedge R(y,x))$	
3	$u \quad \vert\ \exists y (R(u,y) \wedge R(y,u))$	
4	$\quad \vert\ v \quad \vert\ R(u,v) \wedge R(v,u)$	
5	$\quad \vert \quad \vert\ R(u,v)$	$(\wedge\mathrm{E})$, 4
6	$\quad \vert \quad \vert\ R(v,u)$	$(\wedge\mathrm{E})$, 4
7	$\quad \vert \quad \vert\ R(u,v) \to P(u) \wedge \neg P(v)$	$(\forall\mathrm{E})$, 1
8	$\quad \vert \quad \vert\ R(v,u) \to P(v) \wedge \neg P(u)$	$(\forall\mathrm{E})$, 1
9	$\quad \vert \quad \vert\ P(u) \wedge \neg P(v)$	$(\to\mathrm{E})$, 5, 7
10	$\quad \vert \quad \vert\ P(v) \wedge \neg P(u)$	$(\to\mathrm{E})$, 6, 8
11	$\quad \vert \quad \vert\ P(u)$	$(\wedge\mathrm{E})$, 9
12	$\quad \vert \quad \vert\ \neg P(u)$	$(\wedge\mathrm{E})$, 10
13	$\quad \vert \quad \vert\ P(u) \wedge \neg P(u))$	$(\wedge\mathrm{I})$, 11, 12
14	$\quad \vert \quad \vert\ \exists x (P(x) \wedge \neg P(x))$	$(\exists\mathrm{I})$, 13
15	$\quad \vert\ \exists x (P(x) \wedge \neg P(x))$	$(\exists\mathrm{E})$, 3, 4–14
16	$\exists x (P(x) \wedge \neg P(x))$	$(\exists\mathrm{E})$, 2, 3–15

(Q6) The simplest way to do Q6 is to use the derivation in Q5, but change line 13 to $\bot$ (*via* the $(\neg\mathrm{E})$ rule), and then line 14 can be dropped, and lines 15 and 16 also become $\bot$. But notice that the conclusion is "obvious", since "saying" that there is an x so that $P(x)$ and $\neg P(x)$ are both true is "obviously" a contradiction.

(Q7)

1	$\exists z R(z,z)$	
2	$\exists y \forall x S(y,x)$	
3	$u \mid R(u,u)$	
4	$\quad\quad S(u,u)$	
5	$\quad\quad R(u,u)$	(R), 3
6	$\quad S(u,u) \to R(u,u)$	($\to$I), 4–5
7	$\quad \exists z(S(z,u) \to R(u,u))$	($\exists$I), 6
8	$\quad \exists y \exists z(S(z,y) \to R(y,y))$	($\exists$I), 7
9	$\exists y \exists z(S(z,y) \to R(y,y))$	($\exists$E), 1, 2–8

(Q8)

1	$\exists x(P(x) \wedge Q(x))$	
2	$u \mid P(u) \wedge Q(u)$	
3	$\quad P(u)$	($\wedge$E), 2
4	$\quad \exists x P(x)$	($\exists$I), 3
5	$\quad Q(u)$	($\wedge$E), 2
6	$\quad \exists x Q(x)$	($\exists$I), 5
7	$\quad \exists x P(x) \wedge \exists x Q(x)$	($\wedge$I), 4, 6
8	$\exists x P(x) \wedge \exists x Q(x)$	($\exists$I), 1, 2–7

(Q9)

1	$\forall x P(x) \vee \forall x Q(x)$	
2	$u \mid\quad \forall x P(x)$	
3	$\quad\quad P(x)$	($\wedge$E), 2
4	$\quad\quad P(u) \vee Q(u)$	($\vee$I), 3
5	$\quad\quad \forall x Q(x)$	
6	$\quad\quad Q(u)$	($\forall$E), 5
7	$\quad\quad P(u) \vee Q(u)$	($\vee$I), 6
8	$\quad P(u) \vee Q(u)$	($\vee$E), 1, 2–4, 5–7
9	$\forall x(P(x) \vee Q(x))$	($\forall$I), 2–8

(Q10)

1	$\exists x \forall y R(x,y)$	
2	$u \mid v \mid \forall y R(v,y)$	
3	$\quad\quad R(v,u)$	($\forall$E), 2
4	$\quad\quad \exists x R(x,u)$	($\exists$I), 3
5	$\quad \exists x R(x,u)$	($\exists$E), 1, 2–4
6	$\forall y \exists x R(x,y)$	($\forall$I), 2–5

The two problems in Q11 and in Q12 are done similarly: just change the roles of Qx and Qy in each case, for the appropriate quantifier: "$Q = \exists$" in Q11, and "$Q = \forall$" in Q12. Here's half for each.

(Q11)

1	$\exists x \exists y R(x,y)$	
2	$u \mid \exists y R(u,y)$	
3	$\quad v \mid R(u,v)$	
4	$\quad\quad \exists x R(x,v)$	($\exists$I), 3
5	$\quad\quad \exists y \exists x R(x,y)$	($\exists$I), 4
6	$\quad \exists y \exists x R(x,y)$	($\exists$E), 1
7	$\exists y \exists x R(x,y)$	($\exists$E), 1

(Q12)

1	$\forall x \forall y R(x,y)$	
2	$u \mid v \mid \forall y R(u,y)$	($\forall$E), 1
3	$\quad\quad R(u,v)$	($\forall$E), 1
4	$\quad \forall x R(u,x)$	($\forall$I), 2–3
5	$\forall y \forall x R(y,x)$	($\forall$I), 2–4

(Q13)

1		$\neg\exists x(P(x) \wedge Q(x))$	
2	u	$P(u)$	
3		$Q(u)$	
4		$P(u) \wedge Q(u)$	$(\wedge\mathrm{I})$, 2, 3
5		$\exists x(P(x) \wedge Q(x))$	$(\exists\mathrm{I})$, 4
6		$\bot$	$(\neg\mathrm{E})$, 1, 5
7		$\neg Q(u)$	$(\neg\mathrm{I})$, 3–6
8		$P(u) \to \neg Q(u)$	$(\to\mathrm{I})$, 2–7
9		$\forall x(P(x) \to \neg Q(x))$	$(\forall\mathrm{I})$, 2–8

(Q14)

1		$\forall x(P(x) \to \neg Q(x))$	
2		$\exists x(P(x) \wedge Q(x))$	
3	u	$P(u) \wedge Q(u)$	
4		$P(u)$	$(\wedge\mathrm{E})$, 3
5		$Q(u)$	$(\wedge\mathrm{E})$, 3
6		$P(u) \to \neg Q(u)$	$(\forall\mathrm{E})$, 1
7		$\neg Q(u)$	$(\to\mathrm{E})$, 4, 6
8		$\bot$	$(\neg\mathrm{E})$, 5, 7
9		$\bot$	$(\exists\mathrm{E})$, 2, 3–8
10		$\neg\exists x(P(x) \wedge Q(x))$	$(\neg\mathrm{I})$, 2–9

(Q15)

1		$\exists x(P(x) \vee Q(x))$	
2	u	$P(u) \vee Q(u)$	
3		$P(u)$	
4		$\exists x P(x)$	$(\exists\mathrm{I})$, 3
5		$\exists x P(x) \vee \exists x Q(x)$	$(\vee\mathrm{I})$, 4
6		$Q(u)$	
7		$\exists x Q(x)$	$(\exists\mathrm{I})$, 6
8		$\exists x P(x) \vee \exists x Q(x)$	$(\vee\mathrm{I})$, 7
9		$\exists x P(x) \vee \exists x Q(x)$	$(\vee\mathrm{E})$, 2, 3–5, 6–8
10		$\exists x P(x) \vee \exists x Q(x)$	$(\exists\mathrm{I})$, 1, 2–9

(Q16)

1		$\exists x P(x) \vee \exists x Q(x)$	
2		$\exists x P(x)$	
3	u	$P(u)$	
4		$P(u) \vee Q(u)$	$(\vee\mathrm{I})$, 3
5		$\exists x(P(x) \vee Q(x))$	$(\exists\mathrm{I})$, 4
6		$\exists x(P(x) \vee Q(x))$	$(\exists\mathrm{E})$, 2, 3–5
7		$\exists x Q(x)$	
8	v	$Q(v)$	
9		$P(v) \vee Q(v)$	$(\vee\mathrm{I})$, 8
10		$\exists x(P(x) \vee Q(x))$	$(\exists\mathrm{I})$, 9
11		$\exists x(P(x) \vee Q(x))$	$(\exists\mathrm{E})$, 7, 8–10
12		$\exists x(P(x) \vee Q(x))$	$(\exists\mathrm{E})$, 1, 2–6, 7–11

(Q17)

1		$\forall x(P(x) \wedge Q(x))$	
2	u	$P(u) \wedge Q(u)$	$(\forall\mathrm{E})$, 1
3		$P(u)$	$(\wedge\mathrm{E})$, 2
4		$\forall x P(x)$	$(\forall\mathrm{I})$, 2–3
5	v	$P(v) \wedge Q(v)$	$(\forall\mathrm{E})$, 1
6		$Q(v)$	$(\wedge\mathrm{E})$, 5
7		$\forall x Q(x)$	$(\forall\mathrm{I})$, 5–6
8		$\forall x P(x) \wedge \forall x Q(x)$	$(\wedge\mathrm{I})$, 4, 8

(Q18)

1		$\forall x P(x) \wedge \forall x Q(x)$	
2	u	$\forall x P(x)$	$(\wedge\mathrm{E})$, 1
3		$P(u)$	$(\forall\mathrm{E})$, 2
4		$\forall x Q(x)$	$(\wedge\mathrm{E})$, 1
5		$Q(u)$	$(\forall\mathrm{E})$, 4
6		$P(u) \wedge Q(u)$	$(\wedge\mathrm{I})$, 3, 5
7		$\forall x(P(x) \wedge Q(x))$	$(\forall\mathrm{I})$, 2–6

(Q19)

1	$\forall x(P(x) \to Q(x))$	
2	$\exists x(P(x) \land R(x))$	
3	u $P(u) \land R(u)$	
4	$P(u)$	$(\land\text{E})$, 3
5	$R(u)$	$(\land\text{E})$, 3
6	$P(u) \to Q(u)$	$(\forall\text{E})$, 1
7	$Q(u)$	$(\to\text{E})$, 4, 6
8	$Q(u) \land R(u)$	$(\land\text{I})$, 5, 7
9	$\exists x(Q(x) \land R(x))$	$(\exists\text{I})$, 8
10	$\exists x(Q(x) \land R(x))$	$(\exists\text{E})$, 2, 3–9

(Q20)

1	$\forall x(P(x) \lor Q(x))$	
2	$\exists x \neg P(x)$	
3	u $\neg P(u)$	
4	$P(u) \lor Q(u)$	$(\forall\text{E})$, 1
5	$P(u)$	
6	$\bot$	$(\neg\text{E})$, 3, 5
7	$Q(u)$	$(\bot\text{E})$, 6
8	$Q(u)$	
9	$Q(u)$	(R), 8
10	$Q(u)$	$(\lor\text{E})$, 4, 5–7, 8–9
11	$\exists x Q(x)$	$(\exists\text{I})$, 10
12	$\exists x Q(x)$	$(\exists\text{E})$, 2, 3–11

Some remarks:

Q7 Notice that we don't actually need the second premise—(basically because any proposition of the form $P \to \top$ is equivalent to $\top$—in this case, knowing that there is a z so that $R(z,z)$ means that any statement $\exists y(Q \to R(y,y))$ will be "true").

As for the questions (8, 9, & 10) whose converses are invalid, here are some suggestions:

Q8 Notice that the two x's need not be the same in $\exists x P(x) \land \exists x Q(x)$, but they must be the same in $\exists x(P(x) \land Q(x)$, so just think of a situation where they might be different. For example, just because you know there is a blond person and a blue-eyed person in the class, does not guarantee that there is a blond&blue-eyed person there.

Q9 This is similar: for example, it may well be true that everyone in the class is either male or female, but that does not guarantee that everyone is male or that everyone is female.

Q10 This says if there is an x so that $R(x,y)$ holds for all y, then for any y, there is an x (the same x in fact) that satisfies $R(x,y)$, which is "obvious". But the converse says that if for any y there is an x satisfying $R(x,y)$, then one x will work for all the y. This is not always true: you might have different x's for different y's.

For example, consider the positive integers (*i.e.*, the positive whole numbers $1, 2, 3, 4, 5, \ldots$), and let R be "is not smaller than" (*i.e.* "is equal or greater than"): $R(x,y) \equiv$ "$x \geq y$". Then $\forall y \exists x R(x,y)$ is true: it simply says that for any integer, there's a number not smaller than it (*e.g.* the given number itself, or any bigger one). But that need not imply that there is a single "biggest" number, *i.e.* an x so that for every number y, $x \geq y$, for, in fact, there is no "biggest" integer. So $\exists x \forall y R(x,y)$ is false. Similar examples could deal with any situation where there's no single "extreme" instance ("biggest", "smallest", "richest", and so on), even though "locally" there is.

And finally:

Q10–12 Note that a consequence of exercises 10–12 is that you can change the order of quantifiers if they are the same type, but not if they are different types. "$\exists x \exists y \equiv \exists y \exists x$", "$\forall x \forall y \equiv \forall y \forall x$", but "$\exists x \forall y$" is not equivalent to "$\forall y \exists x$". (This is "obvious" in retrospect, if you regard $\exists$ as "being like $\lor$", and $\forall$ as a "being like $\land$", since we know that you can bracket $\lor$'s anyway you want, and likewise $\land$'s, but mixing $\lor$'s and $\land$'s becomes sensitive to brackets. And from this perspective, exercises 8 and 9 fit into the same "story".)