Maths & Logic (360-124)

(Marks) The axioms for a Boolean algebra are the following:

(1a)	$x+y=y+x$	(1b)	$x\cdot y=y\cdot x$	(commutativity)
(2a)	$x+(y+z)=(x+y)+z$	(2b)	$x\cdot(y\cdot z)=(x\cdot y)\cdot z$	(associativity)
(3a)	$x+(y\cdot z)=(x+y)\cdot(x+z)$	(3b)	$x\cdot(y+z)=(x\cdot y)+(x\cdot z)$	(distributivity)
(4a)	$x+(x\cdot y)=x$	(4b)	$x\cdot(x+y)=x$	(absorption)
(5a)	$x+(-x)=1$	(5b)	$x\cdot(-x)=0$	(complements)

(4) 1. Show that the axiom $x+(x\cdot y)=x$ is true in each of the three primary models we have of Boolean algebras, namely informal set theory, classical propositional logic, and the two valued Boolean algebra $\{0,1\}$.

(This means you must verify that $A\cup(A\cap B)=A$, for any sets A,B, as well as $p\vee(p\wedge q)\equiv p$, for any propositions p,q, and also that $x+(x\cdot y)=x$ is true for $x,y\in\{0,1\}$.)

2. Prove the following hold for all elements x,y,z of any Boolean algebra (using the axioms above).

(3×2) (a) $x+0=x$ (b) $x+x=x$ (c) $x\cdot 0=0$

(2×3) (d) If $x+y=1$, then $y=y+(-x)$. (e) $x\cdot y=x$ if and only if $y=x+y$.

(2) 3. Explain, briefly but correctly, in what sense the axioms (5a) and (5b) characterize negation, *i.e.* uniquely determine $-x$ for any element x of a Boolean algebra. You should (must!) quote any technical results proved in class which are necessary for your explanation, but you do not need to prove them.

(4) 4. Prove the following for any group $\langle G,\circ\rangle$:

(a) there is only one unit: if for all elements $a\in G$, $a\circ\iota=\iota\circ a=a$ and $a\circ\varepsilon=\varepsilon\circ a=a$, then $\iota=\varepsilon$;

(b) for every element $a\in G$, there is only one inverse: if $a\circ a'=a'\circ a=\iota$ and $a\circ a''=a''\circ a=\iota$, then $a'=a''$.

(2) 5. Explain why the integers $\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$ with ordinary multiplication do **not** form a group.

(4) 6. In this question, use the basic syntactic types s for sentences and n for nouns.

(a) In the sentences John works and John happily works, John takes the type n, works takes the type $n^r s$, so that John works is a sentence. With these types for John and works, what type must the word happily have, so that John happily works is a sentence? Justify your answer by showing the typing graph for John happily works.

(b) Construct the typing graph for the sentence John likes Jane but she loves him. (Be sure that the type of each word is clearly indicated, and that the type of the whole sentence reduces to s.)

(3) 7. Construct the typing graph for the sentence John loves Jane but he hates her mother. (Be sure that the type of each word is clearly indicated, and that the type of the whole sentence reduces to s.)

(3) 8. Explain briefly why the syntactic types for him and he must be different. Specify what these types are and how they make the necessary distinctions to permit grammatical sentences and to block ungrammatical ones. Use appropriate examples, with typing graphs, to illustrate your claims.

(2) 9. Explain briefly why one might not want the product to be commutative ($xy=yx$) in a pregroup which you want to use to solve the problem of sentence generation.

(4) 10. Write an essay of about 300 words on one of the following. There will be the usual bonus (max 4 extra marks) for doing both.

(a) "Greek mathematics is permanent, more permanent even than Greek literature. Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. 'Immortality' may be a silly word, but probably a mathematician has the best chance of whatever it may mean."

[Godfrey H. Hardy, (1877–1947)]

(b) "There is nothing that can be said by mathematical symbols and relations which cannot also be said by words. The converse, however, is false. Much that can be and is said by words cannot successfully be put into equations, because it is nonsense."

[C. Truesdell,"Six Lectures on Modern Natural Philosophy".]

(Total: 40)