(Marks) **Maths & Logic** (360-124)

(3) 1. Convert the following (base 10) numbers to the indicated bases.
(a) 826 (to base 6) (b) 19473 (to base 8) (c) 725 (to base 4)

(3) 2. Convert the following to base 10:
(a) 1021201_3 (b) 216_7 (c) 1743_9

(2) 3. In base 12, we need digits for the numbers that we call 10 and 11 in the decimal system; use $\mathcal{T}$ and $\mathcal{E}$ for these new digits representing the numbers ten and eleven (so the base 12 digits go $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathcal{T}, \mathcal{E}$). Convert 1734 to base 12. Convert $\mathcal{T}421\mathcal{E}6_{12}$ to base 10.

(3) 4. Define the "divides" predicate $|$, as in "$a \mid b$".
Use the definition to prove $a \mid b$ and $a \mid c$ implies that $a \mid (3c - 2b)$.

(5) 5. Prove using mathematical induction **one** of the following. (Bonus (3) for doing both.)
(a) Prove that for all $n > 0$: $1 + 4 + 7 + \cdots + (3n - 2) = \frac{1}{2}n(3n - 1)$.
(b) Prove that for all $n > 0$: $8 \mid 12^n - 4^n$.

(3) 6. Find **all** the divisors of 168; (hint: there are 16 of them).

(3×2) 7. Find the prime factorization for each of the following numbers.
(a) $5 \cdot 7 \cdot 13 + 1$ (b) 12342 (c) $5! + 17$

(3) 8. List 9 consecutive natural numbers all of which are composite. List another collection of (entirely different) 9 consecutive numbers all of which are composite. *(If you wish, you may leave the numbers in "unevaluated form", as I did in the previous question.)* For each number listed, give at least one non-trivial divisor which justifies the claim that the number is composite.

(3) 9. Prove that 7 is special (*i.e.* that $\forall a, b\ [\ 7 \mid ab\ \rightarrow\ 7 \mid a\ \vee\ 7 \mid b\ ]$).
Prove that 14 is not special.

(4) 10. Use the fact that 7 is special to prove that $\sqrt{7}$ is irrational (*i.e.* that there are no natural numbers m, n for which $\sqrt{7} = m/n$).

((2)) **Bonus:** Although 14 is not special, nonetheless $\sqrt{14}$ is irrational. Modify your proof to show that $\sqrt{14}$ is also irrational; what prevents your new proof from working for $\sqrt{16}$?

(5) 11. Write a brief essay on ONE of the following topics. (Bonus (4) for doing both.)

(a) State precisely the Fundamental Theorem of Arithmetic (you don't have to prove it, just state it correctly).
Explain how the Fundamental Theorem of Arithmetic justifies the claim that any natural number is the product (with appropriate repetition) of all its prime factors. Illustrate this point by defining a suitable number system (with an appropriate notion of multiplication and prime number) in which the Fundamental Theorem of Arithmetic fails; in that system, show an example of a number which is not the product of all its prime factors.

(b) *"Mathematics is often erroneously referred to as the science of common sense. Actually, it may transcend common sense and go beyond either imagination or intuition. It has become a very strange and perhaps frightening subject from the ordinary point of view, but anyone who penetrates into it will find a veritable fairyland, a fairyland which is strange, but makes sense, if not common sense."* (E. Kasner and J. Newman)

We have seen an instance of this when we considered various properties of infinite cardinals. Explain how their properties illustrate this observation, *giving specific technical examples*.

(Total: 40)

Answers

1. (a) 3454_6 (b) 46021_8 (c) 23111_4

2. (a) 937 (b) 111 (c) 1335

3. 1006_{12}, $2\,575\,002$

4. $a \mid b$ means $\exists x(ax = b)$.
   If $a \mid b$, then $ax = b$ for some x, and if $a \mid c$, then $ay = c$ for some y, and so $a(x - 3y) = ax - 3ay = b - 3c$, so $a \mid (b - 3c)$.

5. (a) $1 = \frac{1}{2} \cdot 1 \cdot (3 - 1)$; $1 + 4 + \cdots + (3n - 2) + (3n + 1) = \frac{1}{2}n(3n - 1) + 3n + 1 = \frac{3}{2}n^2 + \frac{5}{2}n + 1 = \frac{1}{2}(n + 1)(3n + 2)$.
   (b) $8|(12 - 4)$; $12^{n+1} - 4^{n+1} = 12 \cdot 12^n - 4 \cdot 4^n = (8 + 4)12^n - 4 \cdot 4^n = 8 \cdot 12^n + 4 \cdot (12^n - 4^n)$ and 8 divides each of these terms.

6. $1, 2, 3, 4, 6, 7, 8, 12, 14, 21, 24, 28, 42, 56, 84, 168$

7. (a) $2^3 \times 3 \times 19$ (b) $2 \times 3 \times 11^2 \times 17$ (c) $5! + 17$ is prime

8. $10!{+}2 = 3\,628\,802, 3\,628\,803, 3\,628\,804, 3\,628\,805, 3\,628\,806, 3\,628\,807, 3\,628\,808, 3\,628\,809, 3\,628\,810$: these are divisible by (respectively) $2, 3, 4, 5, 6, 7, 8, 9, 10$. A smaller list starts with $2 \times 3 \times 5 \times 7 + 2 = 212$: $212, 213, 214, 215, 216, 217, 218, 219, 220$. These are divisible (respectively) by $2, 3, 2, 5, 2, 7, 2, 3, 2$ (for example).

9. Suppose $7 \mid ab$: then consider the prime factorizations ("pf") for a and b. Since the pf of ab contains 7, 7 must be among the primes that occur in the pfs of a and b; in other words, 7 must occur in either the pf for a or for b (or both). If it occurs in the pf for a, then $7 \mid a$; similarly for b.
   14 is not special: Clearly $14 \mid 2 \times 7$, but $14 \nmid 2$ and $14 \nmid 7$.
   Remark: Merely saying that 7 is prime, and primes are special, is not enough — it's exactly that (with 7 as an example) that I am asking you to prove.

10. If $\sqrt{7} = \frac{a}{b}$ (a, b integers with no common factors), then $7 = \frac{a^2}{b^2}$, so $7b^2 = a^2$, *i.e.* $7 \mid a^2$ and so (since 7 is special) $7 \mid a$. So $a = 7x$ for some x, hence $(7x)^2 = a^2 = 7b^2$, and so $7x^2 = b^2$. So $7 \mid b^2$, and so $7 \mid b$, a contradiction.
    Bonus: If we try this with 14, we cannot use the "special" property to deduce $14 \mid a$ from $14 \mid a^2$, since 14 is not special. But since $14 = 2 \times 7$ we can use the property that 2 is special (7 would also do) to get the contradiction (a common factor 2 in a and b). This will not work with 16 however, since $16 \mid a^2$ does not guarantee $16 \mid a$: for example, take $a = 4$. (The "problem" is that 16 does not have an *odd* power of a prime in its prime factorization. This is also the reason why $\sqrt{16}$ is an integer.)

11. As in the text.