Instructor: Dr. R.A.G. Seely

Test 3
(version A)

Maths & Logic (360-124)

The Answers

1.

1			$\forall x\forall y(R(x,y)\to(P(x)\wedge\neg P(y)))$	
2			$\exists x\exists y(R(x,y)\wedge R(y,x))$	
3	u		$\exists y(R(u,y)\wedge R(y,u))$	
4		v	$R(u,v)\wedge R(v,u)$	
5			$R(u,v)$	$(\wedge\mathrm{E})$, 4
6			$R(v,u)$	$(\wedge\mathrm{E})$, 4
7			$\forall y(R(u,y)\to P(u)\wedge\neg P(y))$	$(\forall\mathrm{E})$, 1
8			$R(u,v)\to P(u)\wedge\neg P(v)$	$(\forall\mathrm{E})$, 7
9			$\forall y(R(v,y)\to P(v)\wedge\neg P(y))$	$(\forall\mathrm{E})$, 1
10			$R(v,u)\to P(v)\wedge\neg P(u)$	$(\forall\mathrm{E})$, 9
11			$P(u)\wedge\neg P(v)$	$(\to\mathrm{E})$, 5, 8
12			$P(v)\wedge\neg P(u)$	$(\to\mathrm{E})$, 6, 10
13			$P(u)$	$(\wedge\mathrm{E})$, 11
14			$\neg P(u)$	$(\wedge\mathrm{E})$, 12
15			$P(u)\wedge\neg P(u))$	$(\wedge\mathrm{I})$, 13, 14
16			$\exists x(P(x)\wedge\neg P(x))$	$(\exists\mathrm{I})$, 15
17			$\exists x(P(x)\wedge\neg P(x))$	$(\exists\mathrm{E})$, 3, 4–16
18			$\exists x(P(x)\wedge\neg P(x))$	$(\exists\mathrm{E})$, 2, 3–17

1			$\exists y\forall x A(x,y)$	
2			$\forall x\forall y(A(x,y)\to B(x,y))$	
3	u	v	$\forall x A(x,v)$	
4			$A(u,v)$	$(\forall\mathrm{E})$, 3
5			$A(u,v)\to B(u,v)$	$(\forall\mathrm{E})$, 2
6			$B(u,v)$	$(\to\mathrm{E})$, 4, 5
7			$\exists y B(u,y)$	$(\exists\mathrm{I})$, 6
8			$\exists y B(u,y)$	$(\exists\mathrm{E})$, 1, 3–7
9			$\forall x\exists y B(x,y)$	$(\forall\mathrm{I})$, 3–8

2.

1		$\exists x(P(x) \wedge Q(x))$	
2	u	$P(u) \wedge Q(u)$	
3		$P(u)$	($\forall$E), 2
4		$\exists x P(x)$	($\exists$I), 3
5		$Q(u)$	($\wedge$E), 2
6		$\exists x Q(x)$	($\exists$I), 5
7		$\exists x P(x) \wedge \exists x Q(x)$	($\wedge$I), 4, 6
8		$\exists x P(x) \wedge \exists x Q(x)$	($\exists$I), 1, 2–7

1		$\forall x P(x) \vee \forall x Q(x)$	
2	u	$\forall x P(x)$	
3		$P(u)$	($\wedge$E), 2
4		$P(u) \vee Q(u)$	($\vee$I), 3
5		$\forall x Q(x)$	
6		$Q(u)$	($\forall$E), 5
7		$P(u) \vee Q(u)$	($\vee$I), 6
8		$P(u) \vee Q(u)$	($\vee$E), 1, 2–4, 5–7
9		$\forall x(P(x) \vee Q(x))$	($\forall$I), 2–8

An example why both the reverse arguments are invalid: take as universe all living bodies in my home (humans and cats!). $P(x)$ is "x is human" and $Q(x)$ is "x is feline". Then it's certainly true that there is at least one human and one cat, and for each of us, we are either human or feline. But it's not the case that there is one of us which is both human and feline, and equally, it is not the case that all of us are human or all of us are feline.

3. Two possible translations, with possible answers:

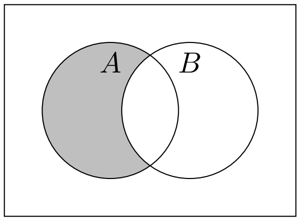
1	$\forall x(\exists y W(y) \to L(x))$	
2	$L(\mathsf{b}) \to T(\mathsf{b})$	
3	$W(\mathsf{a})$	
4	$\exists y W(y) \to L(\mathsf{b})$	($\forall$E), 1
5	$\exists y W(y)$	($\exists$I), 3
6	$L(\mathsf{b})$	($\to$E), 4, 5
7	$T(\mathsf{b})$	($\to$E), 2, 6
8	$\exists x T(x)$	($\exists$I), 7

1	$\exists y W(y) \to \forall x L(x)$	
2	$L(\mathsf{b}) \to T(\mathsf{b})$	
3	$W(\mathsf{a})$	
4	$\exists y W(y)$	($\exists$I), 3
5	$\forall x L(x)$	($\to$E), 1, 4
6	$L(\mathsf{b})$	($\forall$E), 5
7	$T(\mathsf{b})$	($\to$E), 2, 6
8	$\exists x T(x)$	($\exists$I), 7

The point here is that in fact these two WFFs are equivalent. (Exercise: prove this! — Check out Exercise 9 in section 5.6 of the text for one half.)

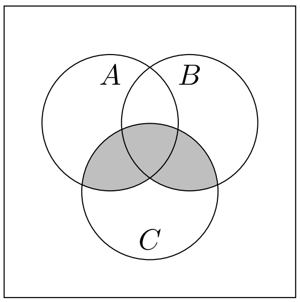
$$\forall x(\exists y W(y) \to L(x)) \quad \leftrightarrow \quad \exists y W(y) \to \forall x L(x)$$

4. $A \setminus (A \cap B) = \{x | x \in A \wedge \neg(x \in A \cap B)\} = \{x | x \in A \wedge \neg(x \in A \wedge x \in B)\} =$
$\{x | x \in A \wedge (x \notin A \vee x \notin B)\} = \{x | (x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B)\} = \{x \in A \wedge x \notin B\} = A \setminus B$.

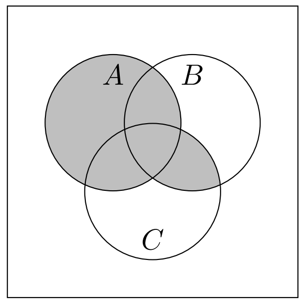


5. (a,b) True (c) False

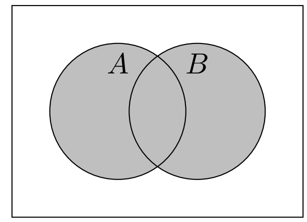
(a)



$\subseteq$

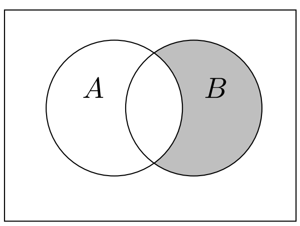


(b)

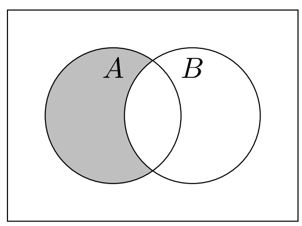


The shaded part equals B only if the part of A outside B is empty.
i.e. if and only if $A \subseteq B$.

(c)



$\neq$



6. $2^3 = 8$ elements of $\mathcal{P}(A)$: $\mathcal{P}(A) = \{\emptyset,\ \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\,\}$
(1 empty set, 3 singletons, 3 pairs, and one triplet).
$2^8 (= 256)$ elements of $\mathcal{P}(\mathcal{P}(A))$, such as
$\emptyset,\quad \{\emptyset\},\quad \{\{a\}\},\quad \{\{b,c\}\},\quad \{\{a,b\},\{b,c\}\},$
$\{\emptyset,\{a\}\},\quad \{\{a\},\{a,b,c\}\},\quad \{\{a,b,c\},\{b,c\},\{b\}\},$
$\{\emptyset,\{a\},\{b,c\}\},\quad \{\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\},\quad \ldots$

7. As discussed in class.
(But it might be good to mention post-modernism and/or relativism ...)