CONTINUOUS LOWER BOUNDS FOR MOMENTS OF ZETA AND L-FUNCTIONS

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In memoriam Professor K. Ramachandra (1933-2011)

Abstract. We obtain lower bounds of the correct order of magnitude for the $2k$-th moment of the Riemann zeta function for all $k \geq 1$. Previously such lower bounds were known only for rational values of k, with the bounds depending on the height of the rational number k. Our new bounds are continuous in k, and thus extend also to the case when k is irrational. The method is a refinement of an approach of Rudnick and Soundararajan, and applies also to moments of L-functions in families.

1. Introduction

Ramachandra [7, 8] established that the moments

$$M_k(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt$$

satisfy the lower bound

$$M_k(T) \geq (c_k + o(1))T(\log T)^{k^2}$$

for natural numbers $2k$. Here c_k is a positive constant, and the estimate holds as T tends to infinity. Ramachandra's result was extended by Heath-Brown [4] to the case when k is any positive rational number. The constant c_k in Heath-Brown's result depends on the height of the rational number k, and does not vary continuously with k. Thus when k is irrational, only the weaker result, due to Ramachandra [9], $M_k(T) \gg T(\log T)^{k^2}(\log\log T)^{-k^2}$ is known. If the truth of the Riemann Hypothesis is assumed, then Ramachandra [8, 10] and independently Heath-Brown [4] showed that it is possible to get the stronger bound $M_k(T) \gg_k T(\log T)^{k^2}$ for all positive real k.

The lower bounds discussed above are essentially of the right order of magnitude for $M_k(T)$. A folklore conjecture states that $M_k(T) \sim C_k T(\log T)^{k^2}$ for a positive constant C_k and all positive real numbers k. This is known for $k = 1$ and 2 (see Chapter VII of [15]), but remains open for all other values. A more precise conjecture, predicting the value for C_k, has been proposed by Keating and Snaith [5]. When $0 \leq k \leq 2$, Heath-Brown [4] showed (assuming RH) that $M_k(T) \ll_k T(\log T)^{k^2}$, and this has recently been extended by Radziwiłł [6] (also

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on RH) to cover the range $k \le 2.18$. For all positive k, Soundararajan [13] has shown (on RH) that $M_k(T) \ll_{k,\epsilon} T(\log T)^{k^2+\epsilon}$.

We now return to the problem of lower bounds for moments which is the focus of this paper. Recently, Rudnick and Soundararajan [11, 12] developed a method for establishing lower bounds of the conjectured order of magnitude for rational moments of L-functions varying in families. As with Heath-Brown's result for $\zeta(s)$, these lower bounds do not vary continuously with the parameter k, and thus one fails to get lower bounds of the right order of magnitude for irrational moments. In this paper, we show how the Rudnick-Soundararajan approach may be extended so as to obtain lower bounds that vary continuously with the moment parameter k. In particular, we thus obtain lower bounds of the right order of magnitude for all real $k \ge 1$. Since this is new already for $\zeta(s)$, we present the proof for that case, and sketch (in section 5) the modifications for L-functions.

Theorem 1. *For any real number $k > 1$, and all large T we have*

$$M_k(T) \ge e^{-30k^4} T(\log T)^{k^2}.$$

We have made no effort to obtain the best possible constant in our Theorem, and with more work it may be possible to obtain a substantially better value. When k is a natural number, Conrey and Ghosh [2] gave the elegant lower bound

$$M_k(T) \ge (1+o(1))T \sum_{n\le T} \frac{d_k(n)^2}{n} \sim T\frac{(\log T)^{k^2}}{\Gamma(k^2+1)} \prod_p \Big(1-\frac{1}{p}\Big)^{k^2}\Big(1+\sum_{a=1}^{\infty}\frac{d_k(p^a)^2}{p^a}\Big),$$

and the best known lower bound (see [14]) is twice as large. These bounds are of the rough shape $M_k(T) \gg k^{-2k^2(1+o_k(1))}T(\log T)^{k^2}$ for large natural numbers k. The constant in the conjectured asymptotic formula for $M_k(T)$ is roughly of size $k^{-k^2(1+o(1))}$ (see, for example, the discussion in [3]).

As in [11], the method presented here applies to any family of L-functions for which slightly more than the first moment can be understood, and produces good lower bounds for all moments larger than the first. There still remains the question of finding lower bounds when $0 < k < 1$. Recently Chandee and Li [1] developed a method for producing good lower bounds for moments of L-functions when $0 < k < 1$ is rational. In forthcoming work, Radziwiłłdescribes an alternative approach which establishes that $M_k(T) \gg_k T(\log T)^{k^2}$ for all real $0 < k < 1$, but his approach is special to the t-aspect and does not adapt to central values of L-functions.

2. Proof of Theorem 1

For any real number $0 < \alpha \le 1$ define the *Sylvester sequence* $s_n = s_n(\alpha)$ as follows: s_1 is the least integer strictly larger than $1/\alpha$, and given $s_1, \ldots, s_n$ take s_{n+1} to be the least integer strictly larger than $(\alpha - \frac{1}{s_1} - \ldots - \frac{1}{s_n})^{-1}$. Thus $\alpha = \sum_{n=1}^{\infty} \frac{1}{s_n(\alpha)}$. When $\alpha = 1$ we obtain the sequence 2, 3, 7, 43, $\ldots$, where the next entry in the sequence is obtained by multiplying the all preceding entries and adding 1, or equivalently by the recurrence

$s_{n+1}(1) = s_n(1)(s_n(1) - 1) + 1$. For any $0 < \alpha \le 1$, we may easily check that $s_1(\alpha) \ge 2$ and $s_{n+1}(\alpha) \ge s_n(\alpha)(s_n(\alpha) - 1) + 1$, so that $s_n(\alpha) \ge s_n(1)$ for all n. The Sylvester sequence grows very rapidly: from the recurrence for $s_{n+1}(1)$ we see easily that $s_n(1) \ge (n-1)! + 1$, and indeed one can show that $s_n(1)$ grows doubly exponentially.

For any $k > 1$, we denote by a_ℓ the Sylvester sequence for $1 - \frac{1}{k}$, and by b_ℓ the Sylvester sequence for 1. Thus $a_\ell \ge b_\ell \ge (\ell - 1)! + 1$, and

$$\sum_{\ell=1}^{\infty} \frac{1}{a_\ell} = 1 - \frac{1}{k}, \qquad \text{and} \qquad \sum_{\ell=1}^{\infty} \frac{1}{b_\ell} = 1.$$

Let T be large, and set $T_0 = T^{1-\vartheta}$, for a small positive parameter ϑ; for example, we may simply take $\theta = 1/100$ below. We define the Dirichlet polynomials

$$A_\ell(s) = \sum_{n \le T_0^{1/a_\ell}} \frac{d_{k/a_\ell}(n)}{n^s}, \qquad \text{and} \qquad B_\ell(s) = \sum_{n \le T_0^{1/b_\ell}} \frac{d_{k/b_\ell}(n)}{n^s}.$$

Let K denote a smooth non-negative function compactly supported inside the interval $[\vartheta, 1-\vartheta]$, with $K(x) \le 1$ for all x, and $K(x) = 1$ for $x \in [2\vartheta, 1 - 2\vartheta]$. For such K we have that the Fourier transform $\hat{K}(\xi) = \int_{-\infty}^{\infty} K(x) e^{-ix\xi} dx$ satisfies for any non-negative integer ν

$$|\hat{K}(\xi)| \ll (1 + |\xi|)^{-\nu}. \tag{1}$$

Our refinement of the method of Rudnick and Soundararajan is based upon a consideration of the quantity

$$\mathcal{I}(T) := \int_0^T K\Big(\frac{t}{T}\Big) \zeta(\tfrac12 + it) \prod_{\ell=1}^{\infty} A_\ell(\tfrac12 + it) B_\ell(\tfrac12 - it) dt. \tag{2}$$

Note that if ℓ is sufficiently large, then $A_\ell(s)$ and $B_\ell(s)$ will be identically 1, and so in the infinite product above only finitely many terms matter. On the one hand, we shall establish a good lower bound for $\mathcal{I}(T)$.

Lemma 1. *With the above notations, we have*

$$\mathcal{I}(T) \ge e^{-15k^3} T (\log T)^{k^2}.$$

On the other hand, by Hölder's inequality we have that bounded by

$$\begin{aligned} \mathcal{I}(T) \;\le\; & \Big(\int_0^T K\Big(\frac{t}{T}\Big) |\zeta(\tfrac12 + it)|^{2k} dt \Big)^{\frac{1}{2k}} \\ & \times \prod_{\ell=1}^{\infty} \Big(\int_0^T K\Big(\frac{t}{T}\Big) |A_\ell(\tfrac12 + it)|^{2a_\ell} dt \Big)^{\frac{1}{2a_\ell}} \Big(\int_0^T K\Big(\frac{t}{T}\Big) |B_\ell(\tfrac12 + it)|^{2b_\ell} dt \Big)^{\frac{1}{2b_\ell}}, \end{aligned}$$

and we may work out upper bounds for the terms in the product above.

Lemma 2. *For any integer $a \geq 1$ we have*

$$\int_0^T K\Big(\frac{t}{T}\Big)\Big|\sum_{n\leq T_0^{1/a}} \frac{d_{k/a}(n)}{n^{\frac12+it}}\Big|^{2a} dt \leq T\sum_{n\leq T_0}\frac{d_k(n)^2}{n} + O(1) \leq T(\log T)^{k^2}.$$

From the above two lemmas and our upper bound on $\mathcal{I}(T)$ we obtain that

$$M_k(T) \geq \frac{\mathcal{I}(T)^{2k}}{(T(\log T)^{k^2})^{(2k-1)}} \geq e^{-30k^4} T(\log T)^{k^2},$$

proving our Theorem.

3. Proof of Lemma 2

Write

$$\Big(\sum_{n\leq T_0^{1/a}} \frac{d_{k/a}(n)}{n^s}\Big)^a \ = \ \sum_{n\leq T_0}\frac{a(n)}{n^s},$$

say, where

$$0 \leq a(n) = \sum_{\substack{n=n_1\ldots n_a\\ n_1,\ldots,n_r\leq T_0^{1/a}}} d_{k/a}(n_1)\cdot\ldots\cdot d_{k/a}(n_a) \ \leq \ d_k(n).$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} K\Big(\frac{t}{T}\Big)\Big|\sum_{n\leq T_0^{1/a}} \frac{d_{k/a}(n)}{n^{\frac12+it}}\Big|^{2a} dt \ &= \ \int_{-\infty}^{\infty} K\Big(\frac{t}{T}\Big)\Big|\sum_{n\leq T_0}\frac{a(n)}{n^{\frac12+it}}\Big|^2 dt\\ &= \ T\sum_{n,m\leq T_0}\frac{a(m)a(n)}{\sqrt{mn}}\hat{K}\Big(T\log\frac{n}{m}\Big).\end{aligned}$$

The diagonal terms $m=n$ above contribute

$$T\hat{K}(0)\sum_{n\leq T_0}\frac{a(n)^2}{n} \leq T\sum_{n\leq T_0}\frac{d_k(n)^2}{n}.$$

To handle the off-diagonal terms $m\neq n$, note that if $m\neq n\leq T_0$ then $|T\log(n/m)| \gg T/T_0 = T^{\vartheta}$ so that by (1) we have $|\hat{K}(T\log(n/m))| \ll T^{-2}$. Therefore the off-diagonal terms contribute

$$\ll T^{-1}\sum_{m,n\leq T_0}\frac{d_k(m)d_k(n)}{\sqrt{mn}} \ll 1.$$

Adding these contributions, we obtain that

$$\mathcal{I}(T) \leq T\sum_{n\leq T_0}\frac{d_k(n)^2}{n} + O(1).$$

Since
$$\sum_{n\le T_0}\frac{d_k(n)^2}{n}\sim\frac{(\log T_0)^{k^2}}{\Gamma(k^2+1)}\prod_p\Big(1-\frac{1}{p}\Big)^{k^2}\Big(1+\sum_{a=1}^{\infty}\frac{d_k(p^a)^2}{p^a}\Big)\le(\log T)^{k^2},$$
the Lemma follows.

4. Proof of Lemma 1

For $s=\frac12+it$ with $\vartheta T\le t\le T$ we have
$$\zeta(s)=\sum_{n\le T}\frac{1}{n^s}+O(T^{-\frac12}).$$
Note that, by Hölder's inequality and Lemma 2,
$$\begin{aligned}&\int_0^T K\Big(\frac{t}{T}\Big)\prod_{\ell=1}^{\infty}|A_\ell(\tfrac12+it)B_\ell(\tfrac12+it)|dt\\ &\le\; T^{\frac{1}{2k}}\prod_{\ell=1}^{\infty}\Big(\int_0^T K\Big(\frac{t}{T}\Big)|A_\ell(\tfrac12+it)|^{2a_\ell}dt\Big)^{\frac{1}{2a_\ell}}\Big(\int_0^T K\Big(\frac{t}{T}\Big)|B_\ell(\tfrac12+it)|^{2b_\ell}\Big)^{\frac{1}{2b_\ell}}\\ &\ll\; T^{1+\epsilon}.\end{aligned}$$
Therefore
$$\mathcal{I}(T)=\int_0^T K\Big(\frac{t}{T}\Big)\sum_{n\le T}\frac{1}{n^{\frac12+it}}\prod_{\ell=1}^{\infty}A_\ell(\tfrac12+it)B_\ell(\tfrac12-it)dt+O(T^{\frac12+\epsilon}).\tag{3}$$

Write
$$\sum_{n\le T}\frac{1}{n^s}\prod_{\ell=1}^{\infty}A_\ell(s)=\sum_n\frac{\alpha(n)}{n^s},\qquad\text{and}\qquad\prod_{\ell=1}^{\infty}B_\ell(s)=\sum_n\frac{\beta(n)}{n^s}.$$
Then both $\alpha(n)$ and $\beta(n)$ are non-negative and bounded above by $d_k(n)$. Moreover $\alpha(n)=0$ if $n>T^2(>TT_0^{\sum_\ell 1/a_\ell})$ and $\beta(n)=0$ if $n>T_0=T_0^{\sum_\ell 1/b_\ell}$. From (3) we obtain that
$$\mathcal{I}(T)=\sum_{m,n}\frac{\alpha(m)\beta(n)}{\sqrt{mn}}T\hat{K}\Big(T\log\frac{n}{m}\Big)+O(T^{\frac12+\epsilon}).$$
If $m\ne n$ and $n\le T_0$ then $|T\log(n/m)|\gg T/T_0=T^{\vartheta}$ and so, using (1), the off-diagonal terms above contribute
$$\ll T^{-2}\sum_{m\le T^2}\sum_{n\le T_0}\frac{d_k(m)d_k(n)}{\sqrt{mn}}\ll 1.$$
Thus
$$\mathcal{I}(T)=T\hat{K}(0)\sum_n\frac{\alpha(n)\beta(n)}{n}+O(T^{\frac12+\epsilon}).\tag{4}$$

Let A_0, A_ℓ, B_ℓ ($\ell \geq 1$) denote parameters all larger than 1, to be chosen later. Set $\alpha_0 = A_0/\log T_0$, and for $\ell \geq 1$, $\alpha_\ell = A_\ell/\log T_0$ and $\beta_\ell = B_\ell/\log T_0$. From the definition, we see that $\alpha(n)\beta(n)$ equals the sum of the quantity $\prod_{\ell\geq 1} d_{k/a_\ell}(m_\ell)d_{k/b_\ell}(n_\ell)$ over all possible ways of writing $n = m_0\prod_{\ell\geq 1} m_\ell = \prod_{\ell \geq 1} n_\ell$ with $m_0 \leq T$, $m_\ell \leq T_0^{1/a_\ell}$ and $n_\ell \leq T_0^{1/b_\ell}$. Next note that

$$m_0^{-\alpha_0}\prod_{\ell\geq 1} m_\ell^{-\alpha_\ell} n_\ell^{-\beta_\ell} - e^{-A_0} - \sum_{\ell\geq 1}\Big(e^{-A_\ell/a_\ell} + e^{-B_\ell/b_\ell}\Big)$$

is always less than 1, and is less than 0 if $m_0 > T(> T_0)$ or if any $m_\ell > T_0^{1/a_\ell}$ or if any $n_\ell > T_0^{1/b_\ell}$. Therefore we see that $\alpha(n)\beta(n)$ is at least as large as

$$\sum_{n=m_0\prod_\ell m_\ell = \prod_\ell n_\ell} \prod_\ell d_{k/a_\ell}(m_\ell)d_{k/b_\ell}(n_\ell)\Big(m_0^{-\alpha_0}\prod_{\ell\geq 1} m_\ell^{-\alpha_\ell}n_\ell^{-\beta_\ell} - e^{-A_0} - \sum_{\ell\geq 1}\Big(e^{-A_\ell/a_\ell}+e^{-B_\ell/b_\ell}\Big)\Big)$$
$$= f(n) - \Big(e^{-A_0} + \sum_{\ell\geq 1}\Big(e^{-A_\ell/a_\ell}+e^{-B_\ell/b_\ell}\Big)\Big)d_k(n)^2,$$

say. Above, we see that $f(n)$ is a multiplicative function of n, and for a prime number p we have (adopting the convention $a_0 = k$)

$$f(p) = \sum_{j=0}^\infty \sum_{\ell=1}^\infty \frac{k^2}{a_j b_\ell} p^{-\alpha_j - \beta_\ell}. \tag{5}$$

Restricting attention to square-free numbers n that are composed only of prime factors below T_0, we see from the above remarks that

$$\sum_n \frac{\alpha(n)\beta(n)}{n} \geq \sum_{p|n \implies p\leq T_0} \frac{\mu(n)^2}{n}\Big(f(n) - \Big(e^{-A_0}+\sum_{\ell\geq 1}\Big(e^{-A_\ell/a_\ell}+e^{-B_\ell/b_\ell}\Big)\Big)d_k(n)^2\Big)$$
$$= \prod_{p\leq T_0}\Big(1+\frac{f(p)}{p}\Big) - \Big(e^{-A_0}+\sum_{\ell\geq 1}\Big(e^{-A_\ell/a_\ell}+e^{-B_\ell/b_\ell}\Big)\Big)\prod_{p\leq T_0}\Big(1+\frac{k^2}{p}\Big).$$

From (5) we see that $f(p) \leq k^2$, and since $(1+y)/(1+x) \geq \exp(y-x)$ whenever $0 \leq y \leq x$, we have $(1 + f(p)/p) \geq (1+k^2/p)\exp((f(p)-k^2)/p)$. Thus we obtain that the above is

$$\geq \prod_{p\leq T_0}\Big(1+\frac{k^2}{p}\Big)\Big(\exp\Big(-\sum_{p\leq T_0}\sum_{j=0}^\infty\sum_{\ell=1}^\infty \frac{k^2}{a_j b_\ell}\Big(\frac1p - \frac{1}{p^{1+\alpha_j+\beta_\ell}}\Big)\Big) - \Big(e^{-A_0}+\sum_{\ell\geq 1}\Big(e^{-A_\ell/a_\ell}+e^{-B_\ell/b_\ell}\Big)\Big)\Big). \tag{6}$$

If x is large and $\alpha > (\log x)^{-\frac12}$ then

$$\sum_{p\leq x}\Big(\frac1p - \frac{1}{p^{1+\alpha}}\Big) \leq \sum_{p\leq x}\Big(\log\Big(1-\frac1p\Big)^{-1} - \log\Big(1-\frac{1}{p^{1+\alpha}}\Big)\Big)$$
$$= \log\log x + \gamma - \log\zeta(1+\alpha) + o(1) \leq \log(\alpha\log x) + \gamma + o(1).$$

On the other hand, if $(\log x)^{-1} < \alpha \le (\log x)^{-\frac{1}{2}}$ then

$$\sum_{p\le x}\Big(\frac{1}{p}-\frac{1}{p^{1+\alpha}}\Big) \le \sum_{p\le e^{1/\alpha}} \frac{\alpha \log p}{p} + \sum_{e^{1/\alpha}\le p\le x} \frac{1}{p} \le 1+\log(\alpha\log x)+o(1).$$

The second bound works for all $\alpha \ge (\log x)^{-1}$, and using it in (6) and substituting that back into (4), we conclude that

$$\mathcal{I}(T) \ge (1+o(1))T\hat{K}(0)\prod_{p\le T_0}\Big(1+\frac{k^2}{p}\Big)$$
$$\times\Big(\exp\Big(-\sum_{j=0}^{\infty}\sum_{\ell=1}^{\infty}\frac{k^2}{a_jb_\ell}(\log(A_j+B_\ell)+1)\Big)-e^{-A_0}-\sum_{\ell=1}^{\infty}(e^{-A_\ell/a_\ell}+e^{-B_\ell/b_\ell})\Big).$$

We now choose $A_0 = 20k^3$, and for $\ell \ge 1$ choose $A_\ell = 20k^3a_\ell^2$ and $B_\ell = 20k^3b_\ell^2$. Then we see easily that

$$e^{-A_0}+\sum_{\ell=1}^{\infty}(e^{-A_\ell/a_\ell}+e^{-B_\ell/b_\ell}) \le 2e^{-20k^3}.$$

Further, using that $\log(a_j^2+b_\ell^2) \le \log(1+a_j^2)+\log(1+b_\ell^2)$,

$$\sum_{j=0}^{\infty}\sum_{\ell=1}^{\infty}\frac{k^2}{a_jb_\ell}(\log(A_j+B_\ell)+1) = k^2+\sum_{\ell=1}^{\infty}\frac{k}{b_\ell}\log(20k^3(1+b_\ell^2))+\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{k^2}{a_jb_\ell}\log(20k^3(a_j^2+b_\ell^2))$$
$$\le k^2\Big(1+\log(20k^3)+\sum_{\ell=1}^{\infty}\frac{\log(1+b_\ell^2)}{b_\ell}+\sum_{j=1}^{\infty}\frac{\log(1+a_j^2)}{a_j}\Big).$$

Now by a calculation we find that $\sum_{\ell=1}^{\infty}\log(1+b_\ell^2)/b_\ell < 5/2$, and since $a_j \ge b_j$ for $j \ge 1$ and $\log(1+x^2)/x$ is decreasing for $x \ge 2$, it also follows that $\sum_{j=1}^{\infty}\log(1+a_j^2)/a_j < 5/2$. Thus we conclude that

$$\sum_{j=0}^{\infty}\sum_{\ell=1}^{\infty}\frac{k^2}{a_jb_\ell}(\log(A_j+B_\ell)+1) \le k^2(6+\log(20k^3)) \le 10k^3.$$

From these estimates, we obtain that for any $k>1$

$$\mathcal{I}(T) \ge (1+o(1))T\hat{K}(0)e^{-12k^3}\prod_{p\le T_0}\Big(1+\frac{k^2}{p}\Big).$$

Restricting attention to $p>k$, and since $(1+x) \ge \exp(x-x^2/2)$ for $0 \le x \le 1$, we have

$$\prod_{p\le T_0}\Big(1+\frac{k^2}{p}\Big) \ge \exp\Big(\sum_{k<p\le T_0}\Big(\frac{k^2}{p}-\frac{k^4}{2p^2}\Big)\Big) \ge \exp\Big(-2k^3+\sum_{p\le T_0}\frac{k^2}{p}\Big) \ge e^{-2k^3}(\log T)^{k^2},$$

since for large T, $\sum_{p\le T_0} 1/p = \log\log T_0 + B_1 + o(1) \ge \log\log T$, where $B_1 = 0.2614\ldots$. Using this in our bound for $\mathcal{I}(T)$, and since $\hat{K}(0) \ge 1-4\theta \ge 3/5$ if $\theta < 1/10$, the Lemma follows at once.

5. Remarks

We may explain the success of our method as follows. Write

$$M_k(T) = \int_0^T \zeta(\tfrac12+it)\prod_{\ell=1}^{\infty}\zeta(\tfrac12+it)^{k/a_\ell}\zeta(\tfrac12-it)^{k/b_\ell}dt.$$

Our idea is then to "approximate" $\zeta(\frac12+it)^{k/a_\ell}$ by $A_\ell(\frac12+it)$ and $\zeta(\frac12-it)^{k/b_\ell}$ by $B_\ell(\frac12-it)$. Note that A_ℓ and B_ℓ are short Dirichlet polynomials, with diminishing length as ℓ increases. This permits the evaluation of the quantity $\mathcal{I}(T)$. On the other hand, we would expect that as ℓ increases, the terms $\zeta(\frac12+it)^{k/a_\ell}\zeta(\frac12-it)^{k/b_\ell}$ make progressively smaller impacts on the moment $M_k(T)$, so that approximating these quantities by shorter Dirichlet polynomials does not entail too great a loss. While the Sylvester sequences seem a natural choice in this construction, all that we require is the convergence of $\sum_\ell \log(1+a_\ell)/a_\ell$ and $\sum_\ell \log(1+b_\ell)/b_\ell$.

We may easily modify this method to the case of L-functions in families. For example, if q is a large prime we may start with

$$\mathcal{I}(q) = \sideset{}{^*}\sum_{\chi \pmod q} L(\tfrac12,\chi)\prod_{\ell=1}^{\infty} A_\ell(\chi)B_\ell(\overline{\chi}),$$

where the sum is over primitive characters χ, and $A_\ell(\chi) = \sum_{n\le q^{\vartheta/a_\ell}} d_{k/a_\ell}(n)\chi(n)/\sqrt{n}$ and $B_\ell(\overline{\chi}) = \sum_{n\le q^{\vartheta/b_\ell}} d_{k/b_\ell}(n)\overline{\chi}(n)/\sqrt{n}$. If ϑ is small, then $\prod_\ell A_\ell(\chi) = \sum_n \alpha(n)\chi(n)/\sqrt{n}$ and $\prod_\ell B_\ell(\overline{\chi}) = \sum_n \beta(n)\overline{\chi}(n)/\sqrt{n}$ are short sums with $\alpha(n)=\beta(n)=0$ if $n\ge q^{\vartheta}$. Therefore $\mathcal{I}(q)$ behaves like a moment of $L(\frac12,\chi)$ slightly larger than the first, and arguing as in [11] we may evaluate $\mathcal{I}(q)$. Then by applying Hölder's inequality we obtain lower bounds for $\sum^*_{\chi \pmod q}|L(\frac12,\chi)|^{2k}$.

To take another example, consider

$$\mathcal{I}(X) = \sideset{}{^\flat}\sum_{|d|\le X} L(\tfrac12,\chi_d)\prod_{\ell=1}^{\infty}A_\ell(\chi_d),$$

where the sum is over fundamental discriminants d, and $A_\ell(\chi_d) = \sum_{n\le X^{\vartheta/a_\ell}} d_{k/a_\ell}(n)\chi_d(n)/\sqrt{n}$. Arguing as in [12] we may evaluate $\mathcal{I}(X)$ for suitably small ϑ, and then obtain a lower bound for $\sum^\flat_{|d|\le X}|L(\frac12,\chi_d)|^k$.

References

[1] V. Chandee and X. Li. *Lower bounds for small fractional moments of Dirichlet L-functions*. Preprint, available at `arXiv:1201.5682`.

[2] J.B. Conrey and A. Ghosh. *Mean values of the Riemann zeta-function*. Mathematika 31 (1984), 159–161.

[3] J.B. Conrey and S. Gonek. *High moments of the Riemann zeta-function*. Duke Math. J. 107 (2001) 577–604.

[4] D.R. Heath-Brown. *Fractional moments of the Riemann zeta-function*. J. London Math. Soc., 24 (1981), 65–78.

[5] J.P. Keating and N.C. Snaith. *Random matrix theory and* $\zeta(1/2+it)$. Comm. Math. Phys. 214 (2000), no. 1, 57–89.

[6] M. Radziwiłł. *The* 4.36*-th moment of the Riemann zeta-function*. IMRN, to appear. Preprint available at `arXiv:1106.4806`.

[7] K. Ramachandra. *Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. I.* Hardy-Ramanujan J., 1 (1978), 1–15.

[8] K. Ramachandra. *Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. II.* Hardy-Ramanujan J., 3 (1980), 1–24.

[9] K. Ramachandra. *Mean value of the Riemann zeta-function and other remarks. III.* Hardy-Ramanujan J., 6 (1983), 1–21.

[10] K. Ramachandra. *Some remarks on the mean value of the Riemann zeta function and other Dirichlet series*. Ann. Acad. Sci. Fennicae, 5 (1980), 145-158

[11] Z. Rudnick and K. Soundararajan. *Lower bounds for moments of L-functions*. Proc. Natl. Acad. Sc. USA 102 (2005) no. 19, 6837-6838.

[12] Z. Rudnick and K. Soundararajan. *Lower bounds for moments of L-functions: symplectic and orthogonal examples*. Multiple Dirichlet series, automorphic forms, and analytic number theory, 293–303. Proc. Sympos. Pure Math. 75, Amer. Math. Soc. (2006).

[13] K. Soundararajan. *Moments of the Riemann zeta-function*. Ann. of Math. 170 (2009) no. 2, 981–993.

[14] K. Soundararajan. *Mean-values of the Riemann zeta-function*. Mathematika 42 (1995), 158–174.

[15] E.C. Titchmarsh. *The theory of the Riemann zeta-function*. Oxford University Press, (1986).

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