HONORS ANALYSIS 3
PRACTICE FINAL

Your final score is your maximum score on 7 questions out of 8. Doing all 8 questions correctly results in bonus points. Each question is worth an equal number of points.

(1) Define
  (a) A measurable function in $\mathbb{R}$
  (b) The maximal function of an integrable $f \in L^1(\mathbb{R})$.

(2) State
  (a) Precisely what $L^1(\mathbb{R})$ completeness means
  (b) The rising sun lemma

(3) Prove that Riemann integrable functions $f : [a, b] \to \mathbb{R}$ are Lebesgue integrable.

(4) Let μ be a measure defined on a σ algebra containing the open sets in $\mathbb{R}$. Suppose that $\mu \ll dx$ where dx is Lebesgue measure. Prove that the function $f(x) = \mu((-\infty, x])$ is absolutely continuous on any finite interval $[a, b]$.

(5) Prove or disprove: If $E \subset \mathbb{R}^2$ is measurable in $\mathbb{R}^2$ then all the slices $E_x = \{y : (x, y) \in E\}$ are measurable in $\mathbb{R}$.

(6) Recall that a collection of balls $\mathcal{B}$ is a *Vitali covering* of a set E if for any $\eta > 0$ and $x \in E$, there is a $B \in \mathcal{B}$ with $\lambda(B) \leq \eta$ and such that $x \in B$. The Vitali covering lemma then states that if $\mathcal{B}$ is a Vitali covering of a measurable set E, then for any δ we can find a sequence of disjoint balls $B_1, \ldots, B_N \in \mathcal{B}$, such that $\lambda(E - \cup_{i=1}^N B_i) \leq 2\delta$. Using this lemma show that if F is absolutely continuous and $F'(x) = 0$ then F is constant.

(7) Let $A \subset [0, 1]$ be a measurable set and $I \subset [0, 1]$ an interval. Find the limit of

$$k \cdot \int_A \int_I (1 - |x - y|)^k dx dy$$

as $k \to \infty$.

(8) Let f_n and g_n be a sequence of integrable functions with $|f_n| \leq |g_n|$. Moreover $g_n \to g$, and $f_n \to f$, and

$$\int_{\mathbb{R}} g_n(x) dx \to \int_{\mathbb{R}} g(x) dx$$

as $n \to \infty$. Prove or disprove:

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)dx.$$