(1) Let $X$ be a set and $\mathcal{M}$ a non-empty collection of subsets of $X$. Prove that if $\mathcal{M}$ is closed under complements and countable unions of disjoint sets, then $\mathcal{M}$ is a $\sigma$-algebra.

(2) Consider the Lebesgue outer measure $\lambda^*$. Prove that a set $E$ is $\lambda^*$-Caratheodory measurable if and only if $E$ is Lebesgue measurable.

(3) Suppose that $\nu, \nu_1, \nu_2$ are signed measures, and $\mu$ is a positive measure on the same $\sigma$-algebra $\mathcal{M}$. Prove
   (a) If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$ then $\nu_1 + \nu_2 \perp \mu$.
   (b) If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$ then $\nu_1 + \nu_2 \ll \mu$.
   (c) $\nu \ll |\nu|$.
   (d) If $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$.

(4) Let $F$ be an increasing function. Recall that we can write $F$ as $F_A + F_C + F_J$ where $F_A$ is absolutely continuous, $F_C$ is continuous with $F'_C = 0$ almost everywhere and $F_J$ is a pure jump function. Given an increasing right continuous function $G$ let $\mu_G$ be the measure such that $\mu_G((a, b]) = G(b) - G(a)$ (this is known as a Borel measure). Let $\mu_A, \mu_C, \mu_J$ denote respectively the measures $\mu_{F_A}, \mu_{F_C}, \mu_{F_J}$. Show that,
   (a) The measure $\mu_A$ is absolutely continuous with respect to Lebesgue measure, and in addition,
   $$\mu_A(E) = \int_E F'(x)dx.$$
   (b) $\mu_C + \mu_J$ and Lebesgue measure are mutually singular.