HW7
DUE NOVEMBER 22TH

(1) Let X be a set and $\mathcal{M}$ a non-empty collection of subsets of X. Prove that if $\mathcal{M}$ is closed under complements and countable unions of disjoint sets, then $\mathcal{M}$ is a σ-algebra.
(2) Consider the Lebesgue outer measure $\lambda^\star$. Prove that a set E is $\lambda^\star$-Caratheodory measurable if and only if E is Lebesgue measurable.
(3) Suppose that ν, ν_1, ν_2 are signed measures, and μ is a positive measure on the same σ-algebra $\mathcal{M}$. Prove
  (a) If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$ then $\nu_1 + \nu_2 \perp \mu$.
  (b) If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$ then $\nu_1 + \nu_2 \ll \mu$.
  (c) $\nu \ll |\nu|$.
  (d) If $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$.
(4) Let F be an increasing function. Recall that we can write F as $F_A + F_C + F_J$ where F_A is absolutely continuous, F_C is continuous with $F_C' = 0$ almost everywhere and F_J is a pure jump function. Given an increasing right continuous function G let μ_G be the measure such that $\mu_G((a, b]) = G(b) - G(a)$ (this is known as a Borel measure). Let μ_A, μ_C, μ_J denote respectively the measures $\mu_{F_A}, \mu_{F_C}, \mu_{F_J}$. Show that,
  (a) The measure μ_A is absolutely continuous with respect to Lebesgue measure, and in addition,
$$\mu_A(E) = \int_E F'(x)dx.$$
  (b) $\mu_C + \mu_J$ and Lebesgue measure are mutually singular.