HW5
DUE OCTOBER 25

(1) Let f, g be integrable functions. Prove that
$$(f \star g)(x) := \int_{\mathbb{R}} f(x-y)g(y)dy$$
is integrable, and that,
$$\|f \star g\|_{L^1} \leq \|f\|_{L^1} \cdot \|g\|_{L^1}.$$
(2) Recall that the Fourier transform of f is defined as
$$\widehat{f}(x) := \int_{\mathbb{R}} f(t)e^{-2\pi ixt}dt.$$
Prove that,
$$\widehat{f \star g}(x) = \widehat{f}(x) \cdot \widehat{g}(x).$$
As an application show that there is no integrable function I such that $f \star I = f$ for all integrable f.
(3) Prove that if f is integrable, and f is not identically zero, then the maximal function $f^\star$ satisfies the inequality,
$$f^\star(x) > \frac{c}{|x|}$$
for some $c > 0$ and all $|x| \geq 1$. Conclude that $f^\star$ is not integrable.
(4) Let E be a fixed set of measure zero. Show that there exists a non-negative integrable f such that,
$$\liminf_{\substack{\lambda(B)\to 0\\ x\in B}} \frac{1}{\lambda(B)} \int_B f(y)dy = \infty$$
for all $x \in E$.
(5) Prove that if E is a measurable subset of $[0,1]$, and there exists an $\alpha > 0$ such that $\lambda(E \cap I) \geq \alpha\lambda(I)$ for all intervals $I \subset [0,1]$ then in fact $\lambda(E) = 1$.