HW 4
DUE OCTOBER 11

(1) Let A be a measurable set with $\lambda(A) > 0$. Show that $A + A = \{x + y : x, y \in A\}$ contains an open interval. (Hint: Consider $f(y) = \int_{\mathbb{R}} \mathbf{1}_A(x)\mathbf{1}_A(y - x)dx$ and its continuity properties)

(2) Let $f : [0, 1] \to \mathbb{R}^+$ be measurable. Suppose that there is a universal constant $C > 0$ such that for all integers $k \geq 1$,

$$\int_0^1 f(x)^k dx = C$$

Prove that there is a measurable set $B \subset [0, 1]$ such that $f(x) = \mathbf{1}_B(x)$ almost everywhere.

(3) Let f be integrable. Prove that there exists a sequence $x_n \to \infty$ such that $x_n|f(x_n)| \to 0$ as $n \to \infty$.

(4) (Riemann-Lebesgue lemma) Let f be integrable, show that,

$$\int_{\mathbb{R}} f(x)\cos(nx)dx \to 0 \ , \ \int_{\mathbb{R}} f(x)\sin(nx)dx \to 0$$

as $n \to \infty$. Alternatively (if you're more comfortable with complex exponentials) show that,

$$\int_{\mathbb{R}} f(x)e^{2\pi inx}dx \to 0.$$

(Hint: Approximate f by step functions)

(5) Prove that given a sequence φ_n and a set of positive measure E, the sequence $\cos(nx+\varphi_n)$ cannot tend to zero as $n \to \infty$, for all $x \in E$. (Hint: Can you first show that the sequence $\cos(nx+\varphi_n)$ cannot tend to a $c \neq 0$? Use the Riemann-Lebesgue lemma)