HW 3 DUE OCTOBER 4

A non-negative measurable function $f \geq 0$ is said to be *integrable* if,

$$\sup_{0 \leq g \leq f} \int_{\mathbb{R}} g(x)dx < \infty.$$

where the supremum is taken over all measurable g with $0 \leq g \leq f$, with g in addition bounded and supported on a set of finite measure. For an integrable f we write,

$$\int_{\mathbb{R}} f(x)dx := \sup_{0 \leq g \leq f} \int_{\mathbb{R}} g(x)dx$$

For general f (not necessarily non-negative) we say that f is integrable if $|f|$ is integrable. Then both $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ are integrable, and we define

$$\int_{\mathbb{R}} f(x)dx := \int_{\mathbb{R}} f^+(x)dx - \int_{\mathbb{R}} f^-(x)dx$$

(1) Suppose that $f \geq 0$ and that f is integrable. If $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$, prove that

$$\lambda(E_\alpha) \leq \frac{1}{\alpha} \int_{\mathbb{R}} f(x)dx.$$

(2) Prove that if f is integrable, and $\int_E f(x)dx \geq 0$ for every measurable E then $f \geq 0$ almost everywhere. Conclude that if $\int_E f(x)dx = 0$ for every measurable E then $f(x) = 0$ almost everywhere.

(3) Show that there exists an integrable f and a sequence of integrable f_n such that

$$\int_{\mathbb{R}} |f(x) - f_n(x)|dx \to 0 \text{ , as } n \to \infty \tag{1}$$

and yet $f_n(x) \to f(x)$ doesn't hold for any $x \in \mathbb{R}$. However show that, for any $\varepsilon > 0$,

$$\lambda(\{x : |f_n(x) - f(x)| > \varepsilon\}) \to 0 \text{ , as } n \to \infty \tag{2}$$

Does (2) imply (1) ?

(4) Consider the function $f(x)$ such that $f(x) = x^{-1/2}$ for $0 < x < 1$ and $f(x) = 0$ otherwise. Let r_n be a fixed enumeration of the rational $\mathbb{Q}$. Consider,

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} \cdot f(x - r_n)$$

Prove that F is integrable, hence the series defining F converges almost everywhere. However, prove also that the series is unbounded on any interval.

(5) Let f be integrable. Let $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that,

$$\int_{\mathbb{R}} |f(x)| dx = \int_0^\infty \lambda(E_\alpha) d\alpha$$