HW 2
HONORS ANALYSIS 3
DUE TUESDAY 27TH SEPTEMBER 2016

(1) A G_δ set is a countable intersection of open sets. Countable unions of closed sets are known as F_δ sets. Prove that a closed set is G_δ set and an open set an F_δ set. Show that there exists F_δ sets which is not G_δ sets.

(2) Let f_n be a sequence of measurable functions on $[0,1]$ with $|f_n(x)| < \infty$ for almost every $x \in [0,1]$. Show that there exists a sequence c_n of positive real numbers such that,

$$\lim_{n\to\infty} \frac{f_n(x)}{c_n} = 0$$

for almost every $x \in [0,1]$. (Hint : Consider c_n for which $\lambda(\{x : |f_n(x)|/c_n > 1/n\}) < 2^{-n}$ and apply Borel-Cantelli)

(3) Show that there is no everywhere continuous function f such that,

$$f(x) = \mathbf{1}_{[0,1]}(x)$$

almost everywhere, and where $\mathbf{1}_{[0,1]}$ is the indicator function of the interval $[0,1]$.

(4) Show that an open disc in $\mathbb{R}^2$ is not a disjoint union of open rectangles. (Hint: What happens to the boundary of any of these rectangles?)

(5) Let f be Lebesgue measurable. Suppose that for all $x, y \in \mathbb{R}$,

$$f(x+y) = f(x) + f(y). \tag{1}$$

Prove that f is continuous. (Hint: You only need to show continuity at zero)

Bonus: Prove that in fact $f(x) = xf(1)$.

(Interestingly, if we don't require f to be measurable, then other solutions to (1) exists, by necessity they are non-measurable).