HONORS ANALYSIS 3
ASSIGNMENT 1
DUE SEPT. 20 IN CLASS

(1) If E is measurable, and $\delta > 0$ is given, define
$$\delta E = \{\delta x : x \in E\}$$
Prove that δE is measurable and that $\lambda(\delta E) = \delta\lambda(E)$.

(2) Suppose that E is a measurable set, and
$$\mathcal{O}_n = \{x : d(x, E) < \frac{1}{n}\}$$
where $d(x, E) = \inf\{|x - y| : y \in E\}$. Prove that if E is compact then $\lambda(E) = \lim_{n\to\infty} \lambda(\mathcal{O}_n)$. Prove also that this is not necessarily true if E is closed and unbounded.

(3) Let A be the subset of $[0, 1]$ in consisting of all real numbers that don't have the digit 7 appearing in their decimal expansion. Show that A is measurable and compute $\lambda(A)$.

(4) Suppose that $A \subset E \subset B$ and A, B are measurable and $\lambda(A) = \lambda(B)$. Prove that E is measurable.

(5) (Borel-Cantelli). Suppose that E_k is a sequence of measurable sets, with
$$\sum_{k=1}^{\infty} \lambda(E_k) < \infty.$$
Let E be the set of x's that belong to infinitely many E_k's. Show that E is measurable. Prove that $\lambda(E) = 0$. (Hint : $E = \cap_{n\geq 1} \cup_{k\geq n} E_k$)

(6) Given an irrational number x, one can find infinitely many reduced fractions p/q with
$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$
(This is known as "Dirichlet's approximation theorem" and is a consequence of the pigeonhole principle). Prove, that for any given $\varepsilon > 0$, the set of x's for which one can find infinitely many reduced fractions p/q with
$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\varepsilon}}$$
is measurable and of Lebesgue measure 0. (Hint: This is an application of Borel-Cantelli)