ZEROS OF MODULAR FORMS IN THIN SETS AND EFFECTIVE QUANTUM UNIQUE ERGODICITY

STEPHEN LESTER, KAISA MATOMÄKI, AND MAKSYM RADZIWIŁŁ

Abstract. We study the distribution of zeros of holomorphic Hecke cusp forms in several "thin" sets as the weight, k, tends to infinity. We obtain unconditional results for slowly shrinking (with k) hyperbolic balls. This relies on a new, effective, proof of Rudnick's theorem and on an effective version of Quantum Unique Ergodicity for holomorphic forms, which we obtain in this paper. In addition, assuming the Generalized Lindelöf Hypothesis or looking only at almost all Hecke cusp forms, we get a formula with a power saving bound for the error term.

We also study the zeros high up in the cusp. Here it is conjectured by Ghosh and Sarnak that these zeros lie on two vertical geodesics. We show that for almost all forms a positive proportion of zeros high up in the cusp are on these geodesics. For all forms, assuming the Generalized Lindelöf Hypothesis, we obtain lower bounds on the number of such zeros which are significantly better than the unconditional results.

1. Introduction

In this article we study zeros of holomorphic cusp forms, f, on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Throughout we will assume that f is an eigenfunction of the Hecke operators. This is natural since the zeros of a sequence of arbitrary cusp forms of weight k do not necessarily equidistribute as $k \to \infty$ (consider for example $f_k(z) = \Delta(z)^{12k}$ with $\Delta(z)$ Ramanujan's delta function). We will assume here that f is normalized so that

$$\iint_{\mathcal{F}} y^k |f(z)|^2 \frac{dxdy}{y^2} = 1$$

where $\mathcal{F} = \{z \in \mathbb{H} : |\operatorname{Re}(z)| \le \frac{1}{2}, |z| > 1\}$ is the usual fundamental domain for $SL_2(\mathbb{Z})\backslash\mathbb{H}$.

Using methods from potential theory, Rudnick [19] showed that the zeros of weight k Hecke cusp forms equidistribute in the fundamental domain $\mathcal{F}$, as $k \to \infty$ with respect to the hyperbolic measure. Rudnick's result was originally conditional on the Quantum Unique Ergodicity (QUE) conjecture for holomorphic Hecke cusp forms

The second author was supported by the Academy of Finland grant no. 137883. The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement n° 320755.

on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. However the latter is now a theorem, proved by Holowinsky and Soundararajan [7], and so Rudnick's result on the equidistribution of zeros holds unconditionally.

It is natural to study what happens beyond equidistribution, and to investigate whether the zeros still equidistribute in sets which are shrinking as $k \to \infty$. Here one faces two immediate obstacles: First of all, it is not clear if it is possible to adapt Rudnick's argument to this setting since it relies on soft techniques which are not immediately applicable to shrinking sets. Secondly, the current results on QUE do not establish a rate of convergence. We remedy the first difficulty by finding a new proof of Rudnick's theorem, which is effective. We address the second difficulty by revisiting the work of Holowinsky and Soundararajan and extracting a rate of convergence from their result. This leads to the following theorem.

Theorem 1.1. *Let f_k be a sequence of Hecke cusp forms of weight k. Also, let $B(z_0, r) \subset \mathcal{F}$ be the hyperbolic ball centered at z_0 and of radius r, with z_0 fixed and $r \geq (\log k)^{-\delta/2+\varepsilon}$ where $\delta = \frac{1}{3} \cdot (33 - 8\sqrt{17}) = 0.005051\ldots$. Then as $k \to \infty$, we have*

$$\frac{\#\{\varrho_f \in B(z_0, r) : f_k(\varrho_f) = 0\}}{\#\{\varrho_f \in \mathcal{F} : f_k(\varrho_f) = 0\}} = \frac{3}{\pi} \iint_{B(z_0,r)} \frac{dxdy}{y^2} + O\Big(r(\log k)^{-\delta/2+\varepsilon}\Big).$$

We also show that the Generalized Lindelöf Hypothesis implies that the zeros of $f(z)$ equidistribute within hyperbolic balls with area as small as $k^{-1/4+\varepsilon}$.

Theorem 1.2. *Assume the Generalized Lindelöf Hypothesis. Let f_k be a sequence of Hecke cusp forms of weight k. Also, let $B(z_0, r) \subset \mathcal{F}$ be the hyperbolic ball centered at z_0 and of radius r, with z_0 fixed and $r \geq k^{-1/8+\varepsilon}$. Then as $k \to \infty$ we have*

$$\frac{\#\{\varrho_f \in B(z_0, r) : f_k(\varrho_f) = 0\}}{\#\{\varrho_f \in \mathcal{F} : f_k(\varrho_f) = 0\}} = \frac{3}{\pi} \iint_{B(z_0,r)} \frac{dxdy}{y^2} + O\Big(rk^{-1/8+\varepsilon}\Big).$$

While QUE establishes that the mass of $y^k|f(z)|^2$ equidistributes as the weight k of f grows, our proof of Theorem 2.1 shows that the equidistribution of the zeros is equivalent to the following much weaker condition: For any fixed $\varepsilon > 0$ and for any fixed domain $\mathcal{R}$, we have

$$\iint_{\mathcal{R}} y^k \cdot |f(z)|^2 \cdot \frac{dxdy}{y^2} \gg e^{-\varepsilon k}.$$

We were not able to make use of this weaker condition, but remain hopeful that it will be useful in later works (see Theorem 2.1 for precise results).

To understand the mass of f in shrinking sets we obtain the following effective version of Quantum Unique Ergodicity in the holomorphic case.

Theorem 1.3 (Effective QUE). *Let f be a Hecke cusp form of weight k. Then,*

$$\sup_{\mathcal{R}\subset\mathcal{F}}\left|\iint_{\mathcal{R}} y^k|f(z)|^2\ \frac{dxdy}{y^2}-\frac{3}{\pi}\iint_{\mathcal{R}}\frac{dxdy}{y^2}\right|\ll_{\varepsilon}(\log k)^{-\delta+\varepsilon}$$

with $\delta=\frac{1}{3}\cdot(33-8\sqrt{17})=0.005051\ldots$ and where the supremum is taken over all the rectangles $\mathcal{R}$ lying inside the fundamental domain $\mathcal{F}$ that have sides parallel to the coordinate axes.

For general domains $\mathcal{R}$ we cannot extract from the argument of Holowinsky and Soundararajan a saving exceeding a small power of $\log k$. However, assuming the Generalized Lindelöf Hypothesis, Watson [24] and Young [26] have established a power saving bound, which is an important ingredient in the proof of Theorem 1.2. On the unconditional front, it was proven by Luo and Sarnak [13, 14] that one can obtain comparable results on average, obtaining a power saving bound for most forms f. Combining this input with our new proof of Rudnick's theorem gives the following variant of Theorem 1.1.

Theorem 1.4. *Let $\mathcal{H}_k$ be a basis for the set of weight k Hecke cusp forms. Let $\delta>0$. Then, for all but at most $\ll k^{20/21+4\delta}$ forms, we have for $r\geq k^{-\delta/2}$*

$$\frac{\#\{\varrho_f\in B(z_0,r):f_k(\varrho_f)=0\}}{\#\{\varrho_f\in\mathcal{F}:f_k(\varrho_f)=0\}}=\frac{3}{\pi}\iint_{B(z_0,r)}\frac{dxdy}{y^2}+O\Big(rk^{-\delta/2}\log k\Big).$$

We are also interested in the distribution of the zeros in shrinking domains in which equidistribution does not happen. It was noticed by Ghosh and Sarnak [1] that the zeros of f high up in the cusp of the fundamental domain, that is the region $\mathcal{F}_Y:=\{z\in\mathcal{F}:\mathrm{Im}(z)>Y\}$ with $Y>\sqrt{k\log k}$, are not expected to equidistribute. This is also a thin set since $\mathrm{Area}_{\mathbb{H}}(\mathcal{F}_Y)=\frac{1}{Y}$ and Ghosh and Sarnak proved that $\frac{k}{Y}\ll\#\{\varrho_f\in\mathcal{F}_Y\}\ll\frac{k}{Y}$. In support of their observation Ghosh and Sarnak showed that many of the zeros of f_k in $\mathcal{F}_Y$ lie on segments of the vertical lines $\mathrm{Re}(z)=0$ and $\mathrm{Re}(z)=-1/2$. They proved that

$$\#\{\varrho_f\in\mathcal{F}_Y:\mathrm{Re}(\varrho_f)=0\text{ or }\mathrm{Re}(\varrho_f)=-1/2\}\gg(k/Y)^{\frac{1}{2}-\frac{1}{40}-\epsilon}. \tag{1.1}$$

The term $1/40$ in their result was subsequently removed in [15] by the second named author. Ghosh and Sarnak conjectured that almost all the zeros of f_k in $\mathcal{F}_Y$ lie on these vertical segments with one half lying on $\mathrm{Re}(z)=0$ and the other half on $\mathrm{Re}(z)=-1/2$ for Y in the range previously specified. We prove two results toward this conjecture. For almost all forms, we establish the following result.

Theorem 1.5. *Let $\varepsilon>0$ be fixed. There exists a subset $\mathcal{S}_k\subset\mathcal{H}_k$, containing more than $(1-\varepsilon)|\mathcal{H}_k|$ elements, and such that every $f\in\mathcal{S}_k$ we have*

$$\#\{\varrho_f\in\mathcal{F}_Y:\mathrm{Re}(\varrho_f)=0\}\geq c(\varepsilon)\cdot\#\{\varrho_f\in\mathcal{F}_Y\}$$

and

$$\#\{\varrho_f \in \mathcal{F}_Y : \mathrm{Re}(\varrho_f) = -1/2\} \geq c(\varepsilon) \cdot \#\{\varrho_f \in \mathcal{F}_Y\}$$

provided that $\delta(\epsilon)k > Y > \sqrt{k \log k}$ *and* $k \to \infty$. *The constants* $\delta(\epsilon)$ *and* $c(\varepsilon)$ *depend only on* ε.

Clearly Theorem 1.5 supports Ghosh and Sarnak's conjecture. The proof of Theorem 1.5 relies on a very recent result on multiplicative functions by the second and third author [16].

For individual forms f we cannot do as well, even on the assumption of the Lindelöf or Riemann Hypothesis. The reason is the following: In order to produce sign changes of f we look at sign changes of the coefficients $\lambda_f(n)$. In order to obtain a positive proportion of the zeros on the line we need a positive proportion of sign changes between the coefficients of $\lambda_f(n)$, in appropriate ranges of n. However we cannot have a positive proportion of sign changes if for example, for all primes $p \leq (\log k)^{2-\varepsilon}$, we have $\lambda_f(p) = 0$. Unfortunately even on the Riemann Hypothesis we cannot currently rule out this scenario.

Nonetheless on the Lindelöf Hypothesis we can still obtain the following result, which is significantly stronger than the previous unconditional result.

Theorem 1.6. *Assume the Generalized Lindelöf Hypothesis. Then for any* $\varepsilon > 0$

$$\#\{\varrho_f \in \mathcal{F}_Y : \mathrm{Re}(\varrho_f) = 0\} \gg (k/Y)^{1-\varepsilon} \tag{1.2}$$

and

$$\#\{\varrho_f \in \mathcal{F}_Y : \mathrm{Re}(\varrho_f) = -1/2\} \gg (k/Y)^{1-\varepsilon}, \tag{1.3}$$

provided that $\sqrt{k \log k} < Y < k^{1-\delta}$ *for some* $\delta > 0$.

The paper is organized as follows: In Section 2 we investigate the results related to equidistribution in shrinking sets. In Section 3 we prove the results on zeros high in the cusp. Finally in Section 4 we establish the effective version of Quantum Unique Ergodicity.

2. Zeros of cusp forms in shrinking geodesic balls

Let ϕ be a smooth function that is compactly supported within $\mathcal{F}$. Also, let $D_r(z)$ be the disk of radius r centered at z. Given a cusp form f, and a compact subset $\mathcal{R} \subset \mathcal{F}$, define,

$$\mu_f(\mathcal{R}) := \iint_{\mathcal{R}} y^k |f(z)|^2 \frac{dxdy}{y^2}.$$

Here the form f is assumed to be normalized so that $\mu_f(\mathcal{F}) = 1$. Also, let $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ denote the hyperbolic Laplacian. The main component of the proofs of Theorems 1.1 is the following:

Theorem 2.1. *Let $\mathcal{R} \subset \{z \in \mathcal{F} : \mathrm{Im}(z) \leq B\}$ where $B > 1$. Also, let $h(k) > 0$ and ϕ be a smooth compactly supported function in $\mathcal{R}$ such that $\Delta\phi \ll h(k)^{-A}$ for some $A \geq 0$. Suppose that f is a Hecke cusp form such that for every $z_0 \in \mathcal{R}$ and $k \geq K(B)$ we have*

$$\mu_f(D_{h(k)}(z_0)) \gg e^{-kh(k)}. \tag{2.1}$$

Then,

$$\begin{aligned}\sum_{\varrho_f} \phi(\varrho_f) =& \frac{k}{12} \cdot \frac{3}{\pi} \iint_{\mathcal{F}} \phi(z) \frac{dxdy}{y^2} + O_B(k \cdot h(k)^2) \\ &+ O_{A,B}\Big(k \cdot h(k) \log 1/h(k) \iint_{\mathcal{F}} |\Delta\phi(z)| \frac{dx\,dy}{y^2}\Big).\end{aligned}$$

By the QUE theorem of Holowinsky and Soundararajan (2.1) holds for fixed, but arbitrarily small $h(k)$. This reproduces the main result of Rudnick [19]. Additionally, Theorem 1.3 implies that (2.1) holds for $h(k) \gg (\log k)^{-\delta+\varepsilon}$ with $\delta = \frac{1}{3} \cdot (33 - 8\sqrt{17}) = 0.005051\ldots$. Assuming the Generalized Lindelöf Hypothesis it follows from an argument of Young [26] that (2.1) holds for $h(k) \geq k^{-1/4+\varepsilon}$. [1]

Proof of Theorems 1.1 and 1.2. Let ϕ_1 be a smooth function such that ϕ_1 has compact support within $B(z_0, r)$, $\phi_1(z) = 1$ for $z \in B(z_0, r - M^{-1})$, and $\Delta\phi_1 \ll M^2$, where M tends to infinity with k and will be chosen later. Also suppose that $r \geq 2/M$. Similarly, let ϕ_2 be a smooth function such that ϕ_2 has compact support within $B(z_0, r + M^{-1},)$, $\phi_2(z) = 1$ for $z \in B(z_0, r)$, and $\Delta\phi_2 \ll M^2$. We have that

$$\begin{aligned}\frac{k}{12} \cdot \frac{3}{\pi} \iint_{\mathcal{F}} |\phi_1(z) - \phi_2(z)| \frac{dxdy}{y^2} \ll& k \cdot \mathrm{Area}_{\mathbb{H}}(B(z_0, r + M^{-1}) \setminus B(z_0, r - M^{-1})) \\ \ll& k \cdot r \cdot M^{-1}.\end{aligned}$$

Also $\iint_{\mathcal{F}} |\Delta\phi_j(z)| \frac{dx\,dy}{y^2} \ll rM$. Next, observe that

$$\sum_{\varrho_f} \phi_1(\varrho_f) \leq \#\{\varrho_f \in B(z_0, r)\} \leq \sum_{\varrho_f} \phi_2(\varrho_f).$$

Thus, Proposition 2.1 implies

$$\#\{\varrho_f \in B(z_0, r)\} = \frac{k}{12} \frac{\mathrm{Area}_{\mathbb{H}}(B(z_0, r))}{\mathrm{Area}_{\mathbb{H}}(\mathcal{F})} + O(rM \cdot k \cdot h(k) \log 1/h(k)) + O(k \cdot r \cdot M^{-1}).$$

[1] In Proposition 5.1 of [26] Young establishes the analog of this for Hecke-Maass cusp forms. The proof for holomorphic case follows in much the same way.

We take $M = h(k)^{-1/2}$, $h(k) = (\log k)^{-\delta+\varepsilon}$, in the unconditional case, and $M = h(k)^{-1/2}, h(k) = k^{-1/4+\varepsilon}$ in the conditional case. Using Theorem 1.3 then completes the proof. The exponent δ is the same exponent as in Theorem 1.3. □

For the proof of Theorem 1.4 we recall the work of Luo and Sarnak [13]. Define the probability measure $\nu := (3/\pi)dxdy/y^2$ and denote by $\mathcal{H}_k$ the space of Hecke cusp forms for the full modular group $SL_2(\mathbb{Z})$. Then, Luo and Sarnak (see Corollary 1.2 in [13]) showed that

$$\frac{1}{\#\mathcal{H}_k} \sum_{f \in \mathcal{H}_k} \sup_B |\mu_f(B) - \nu(B)|^2 \ll k^{-1/21} \tag{2.2}$$

where the supremum is taken over all geodesic balls $B \subset \mathcal{F}$.

Proof of Theorem 1.4. For $r_1 \geq k^{-1/2}$ let

$$\mathcal{E}_k(r_1) := \{f \in \mathcal{H}_k : \exists z_0 \text{ s.t. } B(z_0, r_1) \subset \mathcal{F} \text{ and } \forall z \in B(z_0, r_1),\ y^k |f(z)|^2 \leq k^{-2}\}.$$

Notice that if $f \in \mathcal{H}_k \setminus \mathcal{E}_k(r_1)$ then we may apply Theorem 2.1 with $h(k) \ll r_1$ and argue as in the previous proof to get that for $r \geq \sqrt{r_1}$

$$\#\{\varrho_f \in B(z_0, r)\} = \frac{k}{12} \cdot \frac{\text{Area}_{\mathbb{H}}(B(z_0, r))}{\text{Area}_{\mathbb{H}}(\mathcal{F})} + O(rk \cdot \sqrt{r_1} \log 1/r_1).$$

It remains to bound the size of $\mathcal{E}_k(r_1)$. We apply (2.2) to see that

$$\begin{aligned} r_1^4 \cdot \#\mathcal{E}_k(r_1) &\ll \sum_{f \in \mathcal{E}(r)} \sup\nolimits_{z_0 \in \mathcal{F}} |\mu_f(B(z_0, r_1)) - \nu(B(z_0, r_1))|^2 \\ &\ll \sum_{f \in \mathcal{H}_k} \sup |\mu_f(B) - \nu(B)|^2 \ll k^{20/21}, \end{aligned}$$

where supremum in the second line is over all hyperbolic balls, $B \subset \mathcal{F}$. The claim follows taking $r_1 = k^{-\delta}$. □

2.1. **Proof of Theorem 2.1.** Let ϕ be a smooth function that is compactly supported on $\mathcal{F}$. Our starting point is the following formula of Rudnick (see Lemma 2.1 of [19], note that we assume ϕ is supported in $\mathcal{F}$)

$$\sum_{\varrho_f} \phi(\varrho_f) = \frac{k}{12} \cdot \frac{3}{\pi} \iint_{\mathcal{F}} \phi(z) \frac{dxdy}{y^2} + \frac{1}{2\pi} \iint_{\mathcal{F}} \log(y^{k/2}|f(z)|) \Delta\phi(z) \frac{dxdy}{y^2}. \tag{2.3}$$

To prove Theorem 2.1 we need to bound the second term in the above formula. The difficulty here comes in estimating the contribution to the integral over the set where f is exceptionally small.

We first require the following auxiliary lemma due to Cartan.

Lemma 2.2 (Theorem 9 of [11]). *Given any number $H > 0$ and complex numbers $a_1, a_2, \ldots, a_n$, there is a system of circles in the complex plane, with the sum of the radii equal to $2H$, such that for each point z lying outside these circles one has the inequality*

$$|z - a_1| \cdot |z - a_2| \cdots |z - a_n| > \left(\frac{H}{e}\right)^n.$$

Let $\mathcal{D}$ be the convex hull of $\operatorname{supp} \phi$. Let $\eta, \varepsilon > 0$. We cover $\mathcal{D}$ with N disks of radius ε centered at the points $a_1, \ldots, a_N$. The disks are chosen so that $N \ll \operatorname{Area}(\mathcal{D})/\varepsilon^2$. Define

$$\mathcal{T}_\delta = \{z \in \mathcal{F} : |f(z)y^{k/2}| < e^{-\delta k}\} \qquad \text{and} \qquad \mathcal{T}_{\delta,j} = \mathcal{T}_\delta \bigcap D_\varepsilon(a_j).$$

Let $n_j = \#\{\varrho_f : \varrho_f \in D_{16\varepsilon}(a_j)\}$ and set

$$\mathcal{S}_{\eta,j} = \Big\{ z \in D_\varepsilon(a_j) : \prod_{\varrho_f \in D_{8\varepsilon}(a_j)} |z - \varrho_f| < \left(\frac{\eta\varepsilon}{e}\right)^{n_j} \Big\}.$$

By Cartan's lemma the area of $\mathcal{S}_{\eta,j}$ is $\leq 4\pi\eta^2\varepsilon^2$.

For $z_0 \neq \varrho_f$ define

$$M_r(z_0) := \max_{|z - z_0| \leq r} \left| \frac{f(z)}{f(z_0)} \right|.$$

Lemma 2.3. *Suppose that $\varepsilon > \log k/k$ and $f(z_0)y_0^{k/2} \gg e^{-\varepsilon \cdot k}$. Then there exists a constant $C > 1$ such that*

$$M_{16\varepsilon}(z_0) \ll e^{C\varepsilon \cdot k}.$$

Proof. There is a point $z_{\max} = x_{\max} + iy_{\max}$ such that

$$M_{16\varepsilon}(z_0) = \max_{z \in D_{16\varepsilon}(z_0)} \left| \frac{f(z)}{f(z_0)} \right| = \left| \frac{f(z_{\max})}{f(z_0)} \right| = \left(\frac{y_0}{y_{\max}} \right)^{k/2} \cdot \left| \frac{y_{\max}^{k/2} f(z_{\max})}{y_0^{k/2} f(z_0)} \right|.$$

By Proposition A.1 of Rudnick [19] we have $|y_{\max}^{k/2} f(z_{\max})| \ll k^{1/2}$. (Note that Xia [25] has recently improved this bound to $\ll k^{-1/4+\epsilon}$, but we do not need that here.) Also, $y_0^{k/2} f(z_0) \gg e^{-\varepsilon k}$ and

$$\left(\frac{y_0}{y_{\max}} \right)^{k/2} \leq \left(\frac{y_0}{y_0 - 16\varepsilon} \right)^{k/2} \leq e^{C\varepsilon k}.$$

Combining these bounds we see that

$$M_{16\varepsilon}(z_0) \ll k^{1/2} e^{\varepsilon \cdot k} \cdot e^{C\varepsilon k} \ll e^{C'\varepsilon k}.$$

□

Lemma 2.4. *Suppose $\varepsilon > \log k/k$ and that for all $z_0 \in \mathcal{F}$ we have $\mu_f(D_\varepsilon(z_0)) \gg e^{-\varepsilon k}$. Then there is an absolute constant $\frac{1}{2} > c_0 > 0$ such that for $\delta \geq 1/c_0 \cdot \varepsilon$ we have whenever $\eta > \exp(-c_0\delta/\varepsilon)$ that*

$$\mathcal{T}_{\delta,j} \subset \mathcal{S}_{\eta,j}$$

for each $j = 1, \dots, N$.

Proof. Since $\mu_f(D_\varepsilon(a_j)) \gg e^{-\varepsilon k}$ there exists a point $z_j \in D_\varepsilon(a_j)$ such that $|f(z_j)| \gg e^{-\varepsilon k} y_j^{-k/2}$. If $z \in \mathcal{T}_{\delta,j}$ then

$$\left|\frac{f(z)}{f(z_j)}\right| \ll \left(\frac{y_j}{y}\right)^{k/2} e^{-\delta k+\varepsilon k} \leq \left(\frac{y+2\varepsilon}{y}\right)^{k/2} e^{-\delta k+\varepsilon k} \leq e^{-\delta k+3\varepsilon k} \leq e^{-\delta k/4}. \tag{2.4}$$

Next note that by a lemma of Landau (see equation 3.9.1 of Titchmarsh [23]) if $z_0 \neq \varrho_f$ there is a constant $A > 0$ such that for $|z - z_0| \leq \frac{1}{4}r$

$$\left|\log\frac{f(z)}{f(z_0)} + \sum_{\varrho_f \in D_{r/2}(z_0)} \log\frac{z_0-\varrho_f}{z-\varrho_f}\right| < A \cdot \log M_r(z_0). \tag{2.5}$$

By this and (2.4) we get that for $z \in \mathcal{T}_{\delta,j}$

$$-A\log M_{8\varepsilon}(z_j) < -\delta k/5 + \sum_{\varrho_f \in D_{4\varepsilon}(z_j)} \log\left|\frac{z_j-\varrho_f}{z-\varrho_f}\right|. \tag{2.6}$$

For $z \in D_\varepsilon(a_j) \setminus \mathcal{S}_{\eta,j}$

$$\sum_{\varrho_f \in D_{4\varepsilon}(z_j)} \log\left|\frac{z_j-\varrho_f}{z-\varrho_f}\right| \leq \log \prod_{\varrho_f \in D_{4\varepsilon}(z_j)} \frac{4\varepsilon}{|z-\varrho_f|} \leq n_j \log\frac{4e}{\eta} < A' \log M_{16\varepsilon}(z_j) \log\frac{4e}{\eta}, \tag{2.7}$$

for some absolute constant $A' > 0$ and the last inequality follows from Jensen's formula (we have also used the inequality $\prod_{\varrho_f \in D_{4\varepsilon}(z_j)} |z-\varrho_f| > \prod_{\varrho_f \in D_{8\varepsilon}(a_j)} |z-\varrho_f|$ for $|z - a_j| < \varepsilon$).

For the sake of contradiction, suppose that $\mathcal{T}_{\delta,j}$ is not contained in $\mathcal{S}_{\eta,j}$. Then combining (2.6) and (2.7) it follows that

$$\log M_{16\varepsilon}(z_j) > \frac{\delta k}{5(A + A'\log 4e/\eta)}.$$

However, by Lemma 2.3 $\log M_{16\varepsilon}(z_j) \ll \varepsilon k$, so that a contradiction is reached when c_0 is sufficiently small. $\square$

A simple consequence of the previous lemma gives us a bound on the size of our exceptional set $\mathcal{T}_\delta$. This is one of the main ingredients in the proof of Theorem 2.1.

Observe that under the hypotheses of the previous lemma

$$\operatorname{meas}(\mathcal{T}_\delta \cap \mathcal{D}) \le \sum_{j=1}^{N} \operatorname{meas}(\mathcal{T}_{\delta,j}) \le \sum_{j=1}^{N} \operatorname{meas}(\mathcal{S}_{\eta,j}) \le N 4\pi^2\eta^2\varepsilon^2 \ll \eta^2. \tag{2.8}$$

We also require the following crude, yet sufficient bound on the second moment of $\log y^{k/2}|f(z)|$.

Lemma 2.5. *We have*

$$\iint_{\mathcal{D}} (\log(y^{k/2}|f(z)|))^2 dx\, dy \ll k^2.$$

Proof. We take ε fixed but small and $\delta = 1/c_0 \cdot \varepsilon$. For each $j = 1, 2, \ldots, N$ (note that here $N = O(1)$) there exists $c_j \in D_\varepsilon(a_j)$ such that $c_j \notin \mathcal{S}_{\eta,j}$, which by Lemma 2.4 implies that $c_j \notin \mathcal{T}_{1/c_0\cdot\varepsilon,j}$. Thus, $f(c_j) \gg e^{-1/c_0\cdot\varepsilon k}(\operatorname{Im}(c_j))^{-k/2}$ and $\prod_{\varrho_f \in D_{8\varepsilon}(c_j)} |c_j - \varrho_f| \ge (\varepsilon\eta/e)^{n_j}$.

Now apply (2.5) to see that for $|z - c_j| \le 2\varepsilon$

$$\log\left|\frac{f(z)}{f(c_j)}\right| = \sum_{\varrho_f \in D_{4\varepsilon}(c_j)} \log\left|\frac{z-\varrho_f}{c_j - \varrho_f}\right| + O(\log M_{8\varepsilon}(c_j)).$$

Apply Lemma 2.3 and our earlier observations to see that for $|z - c_j| \le 2\varepsilon$ we have

$$\log|f(z)| = \sum_{\varrho_f \in D_{4\varepsilon}(c_j)} \log\left|z - \varrho_f\right| + O(k).$$

This implies that

$$\int_{|z-a_j|\le\varepsilon} (\log|f(z)|)^2 dz \ll n_j \sum_{\varrho_f \in D_{4\varepsilon}(c_j)} \int_{|z-c_j|\le 2\varepsilon} (\log|z-\varrho_f|)^2 dz + k^2 \ll k^2.$$

Summing over all the disks we see that

$$\iint_{\mathcal{D}} (\log(y^{k/2}|f(z)|))^2 \frac{dx\,dy}{y^2} \ll k^2 \int_{\mathcal{D}} (\log y)^2 dz + \int_{\mathcal{D}} (\log|f(z)|)^2 dz \ll k^2.$$

$\square$

We are now prepared to prove Theorem 2.1.

Proof of Theorem 2.1. By (2.3) it suffices to show that

$$\left|\frac{1}{2\pi}\iint_{\mathcal{F}} \log(y^{k/2}|f(z)|)\Delta\phi(z)\frac{dx\,dy}{y^2}\right| \ll k\cdot h(k)\log 1/h(k)\cdot \iint_{\mathcal{F}} |\Delta\phi(z)|\frac{dx\,dy}{y^2} + k\cdot h(k)^2.$$

Note that for $\delta > \log k/k$

$$\left| \iint_{\mathcal{F}\setminus\mathcal{T}_\delta} \log(y^{k/2}|f(z)|)\Delta\phi(z)\frac{dx\,dy}{y^2}\right| \ll k\delta \iint_{\mathcal{F}} |\Delta\phi(z)|\frac{dx\,dy}{y^2}.$$

Next, note that

$$\left| \iint_{\mathcal{T}_\delta} \log(y^{k/2}|f(z)|)\Delta\phi(z)\frac{dx\,dy}{y^2}\right| \le \left(\iint_{\mathcal{T}_\delta} |\Delta\phi(z)|^2\frac{dx\,dy}{y^2}\right)^{1/2} \cdot \\ \times \left(\iint_{\mathcal{F}} (\log(y^{k/2}|f(z)|))^2\frac{dx\,dy}{y^2}\right)^{1/2}.$$

Since we are assuming $\mu_f(D_{h(k)}(z_0)) \gg e^{-kh(k)}$ equation (2.8) implies that for $\varepsilon \geq h(k)$

$$\iint_{\mathcal{T}_\delta} |\Delta\phi(z)|^2\frac{dx\,dy}{y^2} \ll h(k)^{-2A} \iint_{\mathcal{T}_\delta\cap\mathcal{D}} 1\,\frac{dx\,dy}{y^2} \ll \eta^2 h(k)^{-2A}.$$

Therefore, collecting estimates and applying Lemma 2.5 we have

$$\frac{1}{2\pi}\iint_{\mathcal{F}} \log(y^{k/2}|f(z)|)\Delta\phi(z)\frac{dx\,dy}{y^2} \ll k\delta \iint_{\mathcal{F}} |\Delta\phi(z)|\frac{dx\,dy}{y^2} + k\eta \cdot h(k)^{-A}.$$

We now take $\varepsilon = h(k)$, $\delta = ((A+2)/c_0)\cdot\varepsilon\log 1/\varepsilon$ and $\eta = \exp(-c_0\delta/\varepsilon)$. □

3. Zeros of cusp forms high in the cusp

To detect zeros of f high in the cusp we use the following special case of a result of Ghosh and Sarnak [1, Theorem 3.1] that shows that for certain values of $\mathrm{Im}(z)$ the Hecke cusp form $f(z)$ is essentially determined by one term in its Fourier expansion. In this section we normalize f so that the first term in its Fourier expansion equals one.

Lemma 3.1 (Proposition 2.1 of [15])**.** *There are positive constants c_2, c_3 and δ such that, for all integers $\ell \in (c_2, c_3\sqrt{k/\log k})$ and $f \in \mathcal{H}_k$*

$$\left(\frac{e}{\ell}\right)^{\frac{k-1}{2}} f(x+iy_\ell) = \lambda_f(\ell)e(x\ell) + O(k^{-\delta}),$$

where $y_\ell = \frac{k-1}{4\pi\ell}$.

This essentially tells us that on the vertical geodesic $\mathrm{Re}(z) = 0$ a sign change of $\lambda_f(\ell)$ yields a zero of f. More precisely, to detect a zero on $\mathrm{Re}(z) = 0$ it suffices to find ℓ_1 and ℓ_2 in $(c_2, c_3\sqrt{k/\log k})$ such that

$$\lambda_f(\ell_1) < -k^{-\epsilon} < k^{-\epsilon} < \lambda_f(\ell_2)$$

where $\epsilon > \delta$. A similar analysis holds on the geodesic $\mathrm{Re}(z) = -1/2$, but here one also needs ℓ_1 and ℓ_2 to be odd.

3.1. **Proof of Theorem 1.5.** We detect sign changes for almost all forms using a very recent theorem of the last two authors [16, Theorem 1 with $\delta = (\log h)^{-1/200}$].

Lemma 3.2. *Let $g : \mathbb{N} \to [-1, 1]$ be a multiplicative function. There exists an absolute constant $C > 1$ such that, for any $2 \le h \le X$,*

$$\left| \frac{1}{h} \sum_{x \le n \le x+h} g(n) - \frac{1}{X} \sum_{X \le n \le 2X} g(n) \right| \le (\log h)^{-1/200}$$

for almost all $X \le x \le 2X$ with at most $CX(\log h)^{-1/100}$ exceptions.

To benefit from this, we need to control the number of n for which $|\lambda_f(n)| < n^{-\delta}$ and the number of p for which $\lambda_f(p) < 0$. For this we quote two lemmas. The first one is an immediate consequence of [18, Theorem 2].

Lemma 3.3. *Let p be a prime. Then*

$$\frac{\#\{f \in \mathcal{H}_k \colon |\lambda_f(p)| < p^{-\delta}\}}{\#\mathcal{H}_k} \ll p^{-\delta} + \frac{\log p}{\log k},$$

where the implied constant is absolute and effectively computable.

The second lemma is a large sieve inequality for the Fourier coefficients $\lambda_f(n)$. The version we use is the following special case of a more general theorem [10, Theorem 1] due to Lau and Wu.

Lemma 3.4. *Let $\nu \ge 1$ be a fixed integer. Then*

$$\sum_{f \in \mathcal{H}_k} \left| \sum_{P < p \le Q} \frac{\lambda_f(p^\nu)}{p} \right|^2 \ll_\nu k \frac{1}{P \log P} + k^{10/11} \frac{Q^{\nu/5}}{(\log P)^2}$$

uniformly for

$$2 \mid k, \quad 2 \le P < Q \le 2P.$$

Proof of Theorem 1.5. Let $X = k/Y$. By Lemma 3.1 the claim follows once we have shown that, for all but at most $\varepsilon \cdot \#\mathcal{H}_k$ of $f \in \mathcal{H}_k$, we have, for h large but fixed and for almost all $x \sim X$,

$$\frac{1}{h} \sum_{\substack{x \le n \le x+h \\ |\lambda_f(n)| \ge n^{-\delta}}} |\operatorname{sgn}(\lambda_f(n))| - \frac{1}{h} \left| \sum_{\substack{x \le n \le x+h \\ |\lambda_f(n)| \ge n^{-\delta}}} \operatorname{sgn}(\lambda_f(n)) \right| \gg 1.$$

Lemma 3.2 applied to $g(n) = \operatorname{sgn}(\lambda_f(n))$ and $g(n) = |\operatorname{sgn}(\lambda_f(n))|$ reduces this to showing that, for all but at most $\varepsilon \cdot \#\mathcal{H}_k$ of $f \in \mathcal{H}_k$, we have

$$\frac{1}{X} \sum_{\substack{n \sim X \\ |\lambda_f(n)| \geq n^{-\delta}}} |\operatorname{sgn}(\lambda_f(n))| - \frac{1}{X} \left| \sum_{\substack{n \sim X \\ |\lambda_f(n)| \geq n^{-\delta}}} \operatorname{sgn}(\lambda_f(n)) \right| \gg 1. \tag{3.1}$$

By Lemma 3.3

$$\sum_{f \in \mathcal{H}_k} \sum_{\substack{p \leq X \\ |\lambda_f(p)| < p^{-\delta}}} \frac{1}{p} \ll \#\mathcal{H}_k \cdot \sum_{p \leq X} \left(p^{-1-\delta} + \frac{\log p}{p \log k} \right) = O(\#\mathcal{H}_k).$$

Hence there is an absolute positive constant C such that for given any $\varepsilon > 0$,

$$\sum_{\substack{p \leq X \\ |\lambda_f(p)| < p^{-\delta}}} \frac{1}{p} \leq \frac{C}{\varepsilon} \tag{3.2}$$

for all but at most $\varepsilon/2 \cdot \#\mathcal{H}_k$ forms $f \in \mathcal{H}_k$. Consequently, with this many exceptions,

$$\frac{1}{X} \sum_{\substack{n \sim X \\ |\lambda_f(n)| \geq n^{-\delta}}} 1 = \frac{1}{X} \sum_{\substack{n \sim X \\ |\lambda_f(n)| \geq n^{-\delta}}} |\operatorname{sgn}(\lambda_f(n))| \gg 1. \tag{3.3}$$

On the other hand, since $|\lambda_f(p)| \leq 2$ for all primes p, for any $Q \geq P \geq 2$,

$$\sum_{\substack{P \leq p \leq Q \\ \lambda_f(p) < 0}} \frac{1}{p} \geq \sum_{P \leq p \leq Q} \frac{(\lambda_f(p)^2 - 2\lambda_f(p))}{8p} = \frac{1}{8} \sum_{P \leq p \leq Q} \frac{\lambda_f(p^2) - 2\lambda_f(p) + 1}{p},$$

so that

$$\begin{aligned} \sum_{\substack{p \leq X \\ \lambda_f(p) < 0}} \frac{1}{p} &\geq \sum_{\substack{\log X \leq p \leq X^{1/1000} \\ \lambda_f(p) < 0}} \frac{1}{p} \geq \frac{1}{8} \sum_{\log X \leq p \leq X^{1/1000}} \frac{\lambda_f(p^2) - 2\lambda_f(p) + 1}{p} \\ &= \frac{1 + o(1)}{8} \log \log X + \sum_{\log X \leq p \leq X^{1/1000}} \frac{\lambda_f(p^2) - 2\lambda_f(p)}{p}. \end{aligned}$$

Splitting the last sum into dyadic intervals and then applying Lemma 3.4 we see that it contributes $o(\log \log X)$ for almost all forms f. Hence, recalling (3.2),

$$\sum_{\substack{p \leq X \\ \lambda_f(p) < -p^{-\delta}}} \frac{1}{p} \geq \frac{1 + o(1)}{8} \log \log X$$

for all but $\varepsilon/2\#\mathcal{H}_k$ forms $f \in \mathcal{H}_k$. By Halasz's theorem for real valued functions (see for instance [2]), this implies

$$\frac{1}{X} \sum_{\substack{n \le X \\ 2 \nmid n, |\lambda_f(n)| \ge n^{-\delta}}} \operatorname{sgn}(\lambda_f(n)) = o(1).$$

Hence (3.1) follows from this and (3.3), which completes the proof. □

3.2. **Proof of Theorem 1.6.** Our main proposition for the proof of Theorem 1.6 shows that the Lindelöf hypothesis implies many sign changes of $\lambda_f(\ell)$. For the remainder of this section we use the notation $x \sim X$ to mean $X \le x \le 2X$.

Proposition 3.5. *Assume the Generalized Lindelöf Hypothesis, let $\varepsilon, \eta > 0$ and $X \ge k^\eta$. Then, for almost all $x \sim X$, the interval $[x, x + X^\varepsilon]$ contains integers m_1 and m_2 such that $\lambda_f(m_1) < -X^{-\varepsilon}$ and $\lambda_f(m_2) > X^{-\varepsilon}$.*

In particular, for any $\varepsilon > 0$, there is a sequence $X \le n_1 \le n_2 \le \cdots \le n_M \le 2X$ with $M \gg X^{1-\varepsilon}$ such that

$$(-1)^j \lambda_f(n_j) \ge X^{-\varepsilon}.$$

Observe that the lower bound (1.2), the first part of Theorem 1.6, immediately follows from this along with Lemma 3.1. We will delay the proof of (1.3) until the end of the section.

To prove the proposition, we study first and second moments of $\lambda_f(n)$ in short intervals.

Lemma 3.6. *Assume the Generalized Lindelöf Hypothesis. Let $\epsilon, \eta > 0$, $X \ge k^\eta$, and $2 \le L \le X$. Then*

$$\Big| \sum_{x < n \le x + \frac{x}{L}} \lambda_f(n) \Big| \ll X^\epsilon \Big(\frac{X}{L}\Big)^{1/2}$$

for all $x \sim X$ with at most $X^{1-\epsilon}$ exceptions.

Proof. This follows once we have shown that for any $\varepsilon > 0$

$$\frac{1}{X} \int_X^{2X} \Big| \sum_{x < n \le x + \frac{x}{L}} \lambda_f(n) \Big|^2 dx \ll k^\varepsilon \frac{X}{L^{1-\varepsilon}}. \tag{3.4}$$

We follow an argument of Selberg [20] on primes short intervals. Let $\delta = \log(1+\frac{1}{L}) \approx \frac{1}{L}$. We get by Perron's formula that for $x, x+\frac{x}{L} \notin \mathbb{Z}$

$$\begin{aligned}\sum_{x<n\leq x+\frac{x}{L}} \lambda_f(n) =& \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} L(s,f)\frac{(x+\frac{x}{L})^s - x^s}{s} ds\\ =& x^{1/2}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} L(\tfrac12+it,f)\, w_\delta(\tfrac12+it)\cdot e^{it\cdot\log x} dt,\end{aligned}$$

where $w_\delta(s) = (e^{s\delta}-1)/s$.

Observe that $|w_\delta(s)| \ll \min(\delta/2, 1/|s|)$. Thus, making a change of variables and applying Plancherel we see that

$$\begin{aligned}&\frac{1}{X^2}\int_X^{2X}\Big|\sum_{x<n\leq x+\frac{x}{L}}\lambda_f(n)\Big|^2 dx \leq \int_0^\infty \Big|\sum_{x<n\leq x+\frac{x}{L}}\lambda_f(n)\Big|^2\frac{dx}{x^2}\\ &= \int_{-\infty}^{\infty}\Big|\sum_{e^\tau<n\leq e^{\tau+\delta}}\lambda_f(n)\Big|^2\frac{d\tau}{e^\tau} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}|L(\tfrac12+it,f)|^2|w_\delta(\tfrac12+it)|^2\,dt\\ &\ll k^\varepsilon\Big(\int_{-1/\delta}^{1/\delta}\delta^2|t|^\varepsilon\,dt + \int_{|t|>1/\delta}\frac{1}{|t|^{2-\varepsilon}}\,dt\Big) \ll k^\varepsilon\delta^{1+\varepsilon} \ll \frac{k^\varepsilon}{L^{1-\varepsilon}}.\end{aligned}$$

This establishes (3.4) and the claim follows. □

Lemma 3.7. *Assume the Generalized Lindelöf Hypothesis. Let $\epsilon, \eta > 0$, $X \geq k^\eta$ and $2 \leq L \leq X$. Then*

$$\sum_{x<n\leq x+\frac{x}{L}}|\lambda_f(n)|^2 = \frac{6}{\pi^2}L(1,\operatorname{sym}^2 f)\cdot\frac{x}{L} + O\Big(X^\epsilon\Big(\frac{X}{L}\Big)^{1/2}\Big)$$

for all $x \sim X$ with at most $X^{1-\epsilon}$ exceptions.

Proof. One has

$$\sum_{n\geq 1}\frac{\lambda_f(n)^2}{n^s} = \zeta(2s)^{-1}L(s, f\otimes f) = \frac{\zeta(s)}{\zeta(2s)}L(s,\operatorname{sym}^2 f).$$

Writing $w_\delta(s) = (e^{s\delta} - 1)/s$ and arguing as before only now noting the pole at $s = 1$ we have that

$$\frac{1}{X^2}\int_X^{2X}\left|\sum_{x<n\le x+\frac{x}{L}}\lambda_f(n)^2 - \frac{x}{L}\operatorname{Res}_{s=1}\frac{\zeta(s)L(s,\operatorname{sym}^2 f)}{\zeta(2s)}\right|^2 dx$$
$$\ll \int_{-\infty}^{\infty}\left|\frac{\zeta(\frac12+it)}{\zeta(1+2it)}L(\tfrac12,\operatorname{sym}^2 f)\right|^2 |w_\delta(\tfrac12+it)|^2 dt$$
$$\ll \Bigg(\bigg(\int_{-\infty}^{\infty}(\log(|t|+1))^4\left|\zeta(\tfrac12+it)\right|^4|w_\delta(\tfrac12+it)|^2 dt\bigg)^{1/2}$$
$$\times\bigg(\int_{-\infty}^{\infty}|L(\tfrac12+it,\operatorname{sym}^2 f)|^4|w_\delta(\tfrac12+it)|^2 dt\bigg)^{1/2}\Bigg) \ll \frac{k^\varepsilon}{L^{1-\varepsilon}},$$

and the claim follows. □

Proof of Proposition 3.5. It is shown in [5] that for any $\epsilon > 0$

$$L(1,\operatorname{sym}^2 f) \gg k^{-\epsilon}.$$

Additionally, we have Deligne's bound $\lambda_f(n) \ll n^\epsilon$. Hence, by these facts along with Lemmas 3.6 and 3.7, we have for almost all $x \sim X$ that

$$\sum_{\substack{x\le n\le x+X^\varepsilon\\ \lambda_f(n)\gtrless 0}}\lambda_f(n) \gg X^{9\varepsilon/10}.$$

The claim follows since n with $|\lambda_f(n)| \le X^{-\varepsilon}$ contribute at most $O(1)$ to the sum. □

Proposition 3.8. *Assume the Generalized Lindelöf Hypothesis. Let $\varepsilon,\eta > 0$ and $X \ge k^\eta$. Then, for almost all $x \sim X$, the interval $[x, x+X^\varepsilon]$ contains odd integers m_1 and m_2 such that $\lambda_f(m_1) < -X^{-\varepsilon}$ and $\lambda_f(m_2) > X^{-\varepsilon}$.*

Proof. The proof goes similarly to the proof of Proposition 3.5. Here we have the extra condition $(n,2) = 1$ in the sums. To account for this condition first note that, for $\operatorname{Re}(s) > 1$, $L(s,f)$ and $L(s,\operatorname{sym}^2 f)$ have Euler product representations given in terms of a product of local factors at each prime. That is,

$$L(s,f) = \prod_p L_p(s,f) \qquad \text{and} \qquad L(s,\operatorname{sym}^2 f) = \prod_p L_p(s,\operatorname{sym}^2 f).$$

The argument goes along the same lines as before, except in place of $L(s,f)$ and $L(s,\operatorname{sym}^2 f)$ one uses

$$L(s,f)\cdot(L_2(s,f))^{-1} \qquad \text{and} \qquad L(s,f)\cdot(L_2(s,\operatorname{sym}^2 f))^{-1}.$$

The contribution from the local factor at $p = 2$ is bounded. □

4. Effective QUE

For two smooth, bounded functions h, g the Petersson inner product is given by

$$\langle h, g\rangle = \iint_{\mathcal{F}} h(z)\overline{g(z)}\,\frac{dx\,dy}{y^2}.$$

Let $F_k(z) = y^{k/2} f(z)$ and assume that F_k is normalized so that $\|F_k\|^2 := \langle F_k, F_k\rangle = 1$. In this section we establish QUE with an unconditional, effective error term. Under the assumption of the Generalized Lindelöf Hypothesis effective error terms have been obtained by Watson [24] and Young [26]. For the unconditional result our arguments essentially follow those of Holowinsky and Soundararajan, except for one modification which we have borrowed from Iwaniec's course notes on QUE. We have also used some ideas of Matt Young [26] and the final optimization uses a trick from Iwaniec's course notes on QUE. The following treatment of the inner product of $|F_k|^2$ with a cusp form is taken from Iwaniec's notes on QUE.

Lemma 4.1. *Let u_j be an L^2-normalized Hecke-Maass cusp form with spectral parameter t_j with $|t_j| \le k$. Then,*

$$|\langle |F_k|^2, u_j\rangle| \ll |t_j|^{1/2+\varepsilon}(\log k)^{\varepsilon} \prod_{p\le k}\Big(1 - \frac{n(p)}{p}\Big)$$

where $n(p) = \lambda_f(p^2) + \frac{1}{4}\cdot(1-\lambda_f^2(p^2))$.

Proof. By Watson's formula [24]

$$|\langle u_j F_k, F_k\rangle|^2 \ll \frac{\Lambda(\frac12, u_j \times f \times f)}{\Lambda(1, \mathrm{sym}^2 u_j)\Lambda(1,\mathrm{sym}^2 f)^2}.$$

The ratio of the Gamma factors is $\ll 1/k$, and therefore

$$|\langle u_j F_k, F_k\rangle| \ll \frac{|L(\frac12, u_j \times \mathrm{sym}^2 f)|^{1/2}\cdot |L(\frac12, u_j)|^{1/2}}{\sqrt{k}|L(1,\mathrm{sym}^2 f)|\cdot |L(1,\mathrm{sym}^2 u_j)|^{1/2}}.$$

For the L-functions depending only on u_j we note that the convexity bound gives $|L(\frac12, u_j)| \ll t_j^{1/2+\varepsilon}$, while Corollary 7 of Li [12] implies that $t_j^{-\varepsilon} \ll |L(1,\mathrm{sym}^2 u_j)|$. Next we note that Lemma 2 of Holowinsky and Soundararajan [7] implies

$$|L(1,\mathrm{sym}^2 f)|^{-1} \ll (\log\log k)^3 \prod_{p\le k}\Big(1 - \frac{\lambda_f(p^2)}{p}\Big).$$

Therefore,

$$|\langle u_j F_k, F_k\rangle| \ll (\log\log k)^3 \cdot \frac{t_j^{1/4+\varepsilon}}{\sqrt{k}} \prod_{p\le k}\Big(1 - \frac{\lambda_f(p^2)}{p}\Big)\cdot |L(\tfrac12, u_j\times \mathrm{sym}^2 f)|^{1/2}. \tag{4.1}$$

It suffices to bound the remaining L-function $L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)$. The analytic conductor $\mathfrak{C}$ of $L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)$ satisfies $\mathfrak{C} \asymp (k + |t_j|)^4 \cdot |t_j|^2$. Therefore, by the approximate functional equation (see for instance Theorem 2.1 of Harcos [3]), and then Cauchy-Schwarz,

$$\begin{aligned}|L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)|^2 &\ll \Big(\sum_{n \geq 1} \frac{|\lambda_{u_j}(n) \lambda_f(n^2)|}{\sqrt{n}} \cdot \Big| V\Big(\frac{n}{\sqrt{\mathfrak{C}}}\Big)\Big| \Big)^2 \\ &\ll \sum_{n \geq 1} \frac{|\lambda_{u_j}(n)|^2}{\sqrt{n}} \cdot \Big| V\Big(\frac{n}{\sqrt{\mathfrak{C}}}\Big)\Big| \times \sum_{n \geq 1} \frac{\lambda_f(n^2)^2}{\sqrt{n}} \cdot \Big| V\Big(\frac{n}{\sqrt{\mathfrak{C}}}\Big)\Big|,\end{aligned} \tag{4.2}$$

where V is a smooth function satisfying $|V(x)| \ll_A \min(1, x^{-A})$ for any $A \geq 1$. To bound the second term in (4.2) we use general bounds for multiplicative functions to see

$$\sum_{n \leq \mathfrak{C}^{1/2} (\log \mathfrak{C})^{\varepsilon}} \frac{\lambda_f(n^2)^2}{\sqrt{n}} \ll \mathfrak{C}^{1/4} (\log \mathfrak{C})^{\varepsilon} \prod_{p \leq \mathfrak{C}^{1/2} (\log \mathfrak{C})^{\varepsilon}} \Big(1 + \frac{\lambda_f(p^2)^2 - 1}{p} \Big). \tag{4.3}$$

Next we use Deligne's bound $|\lambda_f(n)| \leq d(n)$, the elementary estimate $\sum_{n \leq X} d^2(n^2) \ll X (\log X)^8$, and partial summation to see that for any $A \geq 1$ that

$$\sum_{n \geq \mathfrak{C}^{1/2} (\log \mathfrak{C})^{\varepsilon}} \frac{\lambda_f(n^2)^2}{\sqrt{n}} \cdot \Big| V\Big(\frac{n}{\sqrt{\mathfrak{C}}}\Big)\Big| \ll_A \frac{\mathfrak{C}^{1/4}}{(\log \mathfrak{C})^A},$$

which is bounded above by the right-hand side of (4.3).

Next observe that for $X \geq 2$

$$\sum_{n \geq 1} \frac{|\lambda_{u_j}(n)|^2}{\sqrt{n}} \cdot e^{-n/X} = \frac{1}{2\pi i} \int_{(2)} \frac{L(\frac{1}{2} + s, u_j \otimes u_j)}{\zeta(2s+1)} \cdot \Gamma(s) X^s ds. \tag{4.4}$$

According to bounds of Kim and Sarnak [9] we have $|\lambda_{u_j}(p)| \leq p^{1/2 - \delta}$ for some $\delta > 0$. Therefore we can apply Heath-Brown's convexity bound [4], which gives

$$|L(\tfrac{1}{2} + it, u_j \otimes u_j)| \ll |t_j|^{1/2} \cdot (1 + |t|)$$

By convexity we also have $|L(\sigma + it, u_j \otimes u_j)| \ll |t_j|^{1/2} \cdot (1 + |t|)$ uniformly in $\sigma \geq \frac{1}{2}$. In addition, from the works Hoffstein and Lockhart [5] and Li [12] we have $|t_j|^{-\varepsilon} \ll L(1, \mathrm{sym}^2 u_j) \ll |t_j|^{\varepsilon}$. Combining these ingredients it follows that (4.4) equals

$$\frac{6}{\pi^{3/2}} X^{1/2} L(1, \mathrm{sym}^2 \ u_j) + O\Big(X^{\varepsilon} \cdot \sqrt{|t_j|} \Big).$$

Using this and partial summation it follows that the first term on the right-hand side of (4.2) is $\ll \mathfrak{C}^{1/4} L(1, \mathrm{sym}^2 u_j) + \mathfrak{C}^{\varepsilon} \sqrt{|t_j|}$. Thus, applying this bound along with

(4.3) in (4.2) yields

$$|L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)|^{1/2} \ll (\log \mathfrak{C})^{\varepsilon} \Big(\prod_{p \leq \mathfrak{C}^{1/2}(\log \mathfrak{C})^{\varepsilon}} \Big(1 + \frac{\lambda_f(p^2)^2 - 1}{4p}\Big)\Big) \Big(\mathfrak{C}^{1/8}|t_j|^{\varepsilon} + \mathfrak{C}^{1/16+\varepsilon}|t_j|^{1/8}\Big).$$

Using this in (4.1), doing some minor manipulations in the Euler products, and simplifying error terms we have that

$$|\langle u_j F_k, F_k\rangle| \ll |t_j|^{1/2+\varepsilon} (\log k)^{\varepsilon} \prod_{p \leq k} \Big(1 - \frac{n(p)}{p}\Big)$$

as claimed.

□

We will also require the following bound for the inner products of Maass cusp forms u_j and Eisenstein series $E(\cdot, \frac{1}{2} + it)$ with a smooth function ϕ.

Lemma 4.2. *Let ϕ be a smooth function with support contained within $\{z : 1/2 \leq \mathrm{Im}(z) \leq C\}$ with $C > 1$. Also, suppose ϕ satisfies $\Delta^{\ell}\phi \ll M^{2\ell}$ for all $\ell \geq 1$. Then,*

$$|\langle u_j, \phi\rangle| \ll_A \frac{M^{2A}}{1 + |t_j|^{2A - \frac{1}{2}}} \quad \textit{and} \quad |\langle E(\cdot, \tfrac{1}{2} + it), \phi\rangle| \ll_A \sqrt{C}\frac{M^{2A}}{1 + |t|^{2A-4}}.$$

Proof. The hyperbolic Laplacian is symmetric with respect to the Petersson inner product, that is, $\langle \Delta g, h\rangle = \langle g, \Delta h\rangle$. Therefore since u_j is an eigenfunction of Δ with eigenvalue $\frac{1}{4} + t_j^2$, we get

$$(\tfrac{1}{4} + t_j^2)^{\ell}\langle u_j, \phi\rangle = \langle \Delta^{\ell} u_j, \phi\rangle = \langle u_j, \Delta^{\ell}\phi\rangle \ll \|u_j\|_{\infty} \cdot M^{2\ell}.$$

Since $\|u_j\|_{\infty} \ll |t_j|^{1/2}$ this gives the first claim. For the second claim we proceed in the same way, finding that

$$(\tfrac{1}{4} + t^2)^{\ell}\langle E(\cdot, \tfrac{1}{2} + it), \phi\rangle = \langle E(\cdot, \tfrac{1}{2} + it), \Delta^{\ell}\phi\rangle \ll M^{2\ell} \sup_{z} |E(z, \tfrac{1}{2} + it)|,$$

where the supremum is taken over $z \in \mathcal{F}$ such that $\mathrm{Im}(z) \leq C$. In this range we can bound the Eisenstein series by $\ll \sqrt{C}|t|^4$. □

We now derive estimates for $\langle |F_k|^2, \phi\rangle$ in two different ways. The first approach, below, is obtained using the spectral decomposition, and follows Soundararajan's paper, except for the use of Lemma 4.1.

Lemma 4.3. *Let ϕ be as in Lemma 4.2. If f is a Hecke cusp form of weight k, and $M \le \log k$, then*

$$\langle |F_k|^2, \phi\rangle = \frac{3}{\pi}\cdot\langle 1,\phi\rangle + O\Big(M^{3/2+\varepsilon}(\log k)^{\varepsilon}\cdot\Big(\prod_{p\le k}\Big(1-\frac{n(p)}{p}\Big)+\prod_{p\le k}\Big(1-\frac{\lambda_f(p^2)+1}{p}\Big)\Big)\|\phi\|_2\Big)$$
$$+ O_A(\sqrt{C}(\log k)^{-A})$$

and where $n(p) = \lambda_f(p^2) + \frac{1}{4}\cdot(1-\lambda_f(p^2)^2)$.

Proof. Starting with the spectral decomposition we have (see for instance Theorem 15.5 of [8])
(4.5)
$$\langle |F_k|^2,\phi\rangle = \frac{3}{\pi}\cdot\langle 1,\phi\rangle + \sum_{j\ge 1}\langle |F_k|^2, u_j\rangle\langle u_j,\phi\rangle + \frac{1}{4\pi}\int_{\mathbb{R}}\langle |F_k|^2, E(\cdot,\tfrac{1}{2}+it)\rangle\langle E(\cdot,\tfrac{1}{2}+it),\phi\rangle dt.$$

By the previous lemma we have

$$|\langle u_j,\phi\rangle| \ll_A \frac{M^{2A}}{1+|t_j|^{2A-\frac{1}{2}}} \quad\text{and}\quad |\langle E(\cdot,\tfrac{1}{2}+it),\phi\rangle| \ll_A \sqrt{C}\frac{M^{2A}}{1+|t|^{2A-4}}$$

for any fixed $A > 0$.

Combining Corollary 1 of [22] with Lemma 2 in [7] we have

$$|\langle |F_k|^2, E(\cdot,\tfrac{1}{2}+it)\rangle| \ll (1+|t|)\exp\Big(-\sum_{p\le k}\frac{\lambda_f(p^2)+1}{p}\Big)(\log k)^{\varepsilon}. \tag{4.6}$$

(Note that here we have used a slightly stronger form of Corollary 1 of [22], which is easily seen to follow from the proof.) Using the above bounds with Lemma 4.1 it follows that the terms with $|t_j| > M(\log k)^{\varepsilon}$ and $|t| > M(\log k)^{\varepsilon}$ in (4.5) contribute an amount of at most $O(\sqrt{C}(\log k)^{-A})$. Recalling Weyl's law, that is $\sum_{|t_j|\le T} 1 \sim T^2/12$, which has been established here by Selberg, we see that the contribution of the remaining cusp forms is bounded by

$$\begin{aligned}(4.7)\quad &\Big(\sum_{|t_j|\le M(\log k)^{\varepsilon}}|\langle |F_k|^2,u_j\rangle|^2\Big)^{1/2}\cdot\Big(\sum_j |\langle u_j,\phi\rangle|^2\Big)^{1/2} \\ &\qquad \ll M^{3/2}(\log k)^{\varepsilon}\cdot\prod_{p\le k}\Big(1-\frac{n(p)}{p}\Big)\cdot\|\phi\|_2\end{aligned}$$

using Lemma 4.1 and Bessel's inequality. On the other hand the contribution of the remaining Eisenstein series is bounded by

$$
\begin{aligned}
(4.8)\qquad & \Big(\int_{|t| \le M(\log k)^{\varepsilon}} |\langle |F_k|^2, E(\cdot, \tfrac12 + it)\rangle|^2 \, dt \Big)^{1/2} \cdot \Big(\int_{\mathbb{R}} |\langle E(\cdot, \tfrac12 + it), \phi\rangle|^2 \, dt \Big)^{1/2} \\
& \ll M^{3/2} (\log k)^{\varepsilon} \cdot \prod_{p \le k} \Big(1 - \frac{\lambda_f(p^2) + 1}{p} \Big) \cdot \|\phi\|_2
\end{aligned}
$$

using equation (4.6) and Bessel's inequality. Using (4.7) and (4.8) in (4.5) gives the claim. □

The second lemma obtains an estimate for $\langle |F_k|^2, \phi\rangle$ by using a incomplete Poincare series and bounds for a shifted convolution problem. This mirrors the route taken by Holowinsky.

Lemma 4.4. *Let h be a smooth function, such that $x^j h^{(\ell)}(x) \ll M^{\ell}$ for all integers $j, \ell \ge 0$ and assume $M \le \log k$. Suppose in addition that h is supported in $[1/2, C]$ with $C \le \log k$. Let $I \subset [-1/2, 1/2]$ an interval. Then,*

$$
\begin{aligned}
\langle |F_k|^2, h(y)\chi_I(x)\rangle = \frac{3}{\pi} \int_0^{\infty} \int_I h(y) \, \frac{dxdy}{y^2} + O\Big(C (\log k)^{\varepsilon} \prod_{p \le k} \Big(1 - \frac{(|\lambda_f(p)| - 1)^2}{p} \Big)\Big) \\
+ O(M^3 (\log k)^{-1+\varepsilon}).
\end{aligned}
$$

Proof. Consider the following incomplete Poincare series,

$$
P_{h,m}(z) := \sum_{\gamma \in \Gamma \backslash \Gamma_\infty} h(\mathrm{Im}(\gamma z)) e(m \, \mathrm{Re}(\gamma z)).
$$

Then, using the standard unfolding method, we get

$$
(4.9)\qquad \langle |F_k|^2, P_{h,m}(z)\rangle = \int_{-1/2}^{1/2} \int_0^{\infty} |F_k(z)|^2 \cdot h(y) e(mx) \cdot \frac{dxdy}{y^2}.
$$

Applying Proposition 2.1 of [14], which follows from expanding $|F_k|^2$, and keeping track of the dependencies on m and h one has

$$
\begin{aligned}
(4.10)\qquad \langle |F_k|^2, P_{h,m}(z)\rangle = & \frac{2\pi^2}{(k-1) L(1, \mathrm{sym}^2 f)} \sum_{r \ge 1} \lambda_f(r) \lambda_f(r+m) h\Big(\frac{k-1}{4\pi(r + m/2)} \Big) \\
& + O\Big(\frac{(|m| + M)^B}{k^{1-\varepsilon}} \Big),
\end{aligned}
$$

where B is a sufficiently large absolute constant.

Using Beurling-Selberg polynomials (see for instance Chapter 1 of Montgomery [17]) there exists coefficients $a^-_{\ell,H}(I)$ and $a^+_{\ell,H}(I)$ such that $|a^{\pm}_{\ell,H}(I)| \ll 1/\ell$ and

$$(4.11)\quad |I| - \frac{1}{H+1} + \sum_{0\neq|\ell|\leq H} a^-_{\ell,H}(I)e(\ell x) \leq \chi_I(x) \leq |I| + \frac{1}{H+1} + \sum_{0\neq|\ell|\leq H} a^+_{\ell,H}(I)e(\ell x).$$

Combining (4.9), (4.10), and (4.11) it follows that $\langle |F_k|^2, h(y)\chi_I(x)\rangle$ equals

(4.12)

$$\frac{2\pi^2}{(k-1)L(1,\mathrm{sym}^2 f)}\Bigg(\Big(|I| + O\Big(\frac{1}{H}\Big)\Big)\sum_{r\geq 1}|\lambda_f(r)|^2\cdot h\Big(\frac{k-1}{4\pi r}\Big)$$
$$+ O\Big(\sum_{0\neq|m|\leq H}\frac{1}{m}\sum_{r\geq 1}|\lambda_f(r)\lambda_f(r+m)|\cdot h\Big(\frac{k-1}{4\pi(r+\frac{m}{2})}\Big)\Big)\Bigg) + O\Big(\frac{H(H+M)^B}{k^{1-\varepsilon}}\Big)\Big).$$

We now evaluate the main term in (4.12) using Soundararajan's [22] weak sub-convexity estimate. Note that by Mellin inversion,

$$h(x) = \frac{1}{2\pi i}\int_{(2)} G(s)x^s ds\ ,\ G(s) := \int_0^\infty h(1/y)y^{s-1}dy.$$

In particular $G(s)$ is entire and $G(s) \ll_A M^A\cdot(1+|s|)^{-A}$ for any fixed A in any vertical strip $0 < a \leq \mathrm{Re}(s) \leq b$. We express the main term as

$$\sum_{r\geq 1}|\lambda_f(r)|^2\cdot h\Big(\frac{k-1}{4\pi r}\Big) = \frac{1}{2\pi i}\int_{(2)}\Big(\frac{k-1}{4\pi}\Big)^s\cdot\frac{L(s,f\otimes f)}{\zeta(2s)}\cdot G(s)\ ds.$$

Shifting contours to $\mathrm{Re}(s) = \frac{1}{2}$ we collect a pole at $s=1$ with residue

$$\frac{k-1}{4\pi}\cdot\frac{6}{\pi^2}\int_0^\infty h(y)\frac{dy}{y^2}\cdot L(1,\mathrm{sym}^2 f).$$

To bound the integral on the line $\mathrm{Re}(s) = \frac{1}{2}$ use the following weak-sub-convexity bound due to Soundararajan [22] on $\mathrm{Re}(s) = \frac{1}{2}$

$$|L(\tfrac{1}{2}+it,\mathrm{sym}^2 f)| \ll_\varepsilon \frac{k^{1/2}(1+|t|)}{(\log k)^{1-\varepsilon}}.$$

As a result we conclude that

$$(4.13)\quad \sum_{r\geq 1}|\lambda_f(r)|^2\cdot h\Big(\frac{k-1}{4\pi r}\Big) = \frac{k-1}{4\pi}\cdot\frac{6}{\pi^2}\int_0^\infty h(y)\frac{dy}{y^2}\cdot L(1,\mathrm{sym}^2 f) + O\Big(\frac{M^3\cdot k}{(\log k)^{1-\varepsilon}}\Big).$$

To handle the off-diagonal terms in (4.12) we use a version of Shiu's bound (as in Holowinsky's work [6]). Using this bound we see that the off-diagonal is

$$\ll Ck(\log k)^{\varepsilon} \prod_{p \le k} \Big(1 + \frac{2|\lambda_f(p)| - 2}{p}\Big) \cdot (\log H)^2, \tag{4.14}$$

where we have bounded the terms in the Euler product with $k < p \le Ck(\log k)^{\varepsilon}$ by $O(1)$ (here we use the bound $C < k$). Take $H = (\log k)^3$. Applying (4.13) and (4.14) in (4.12) and using the bounds

$$(\log\log k)^{-3} \exp\Big(\sum_{p \le k} \frac{\lambda_f(p^2)}{p}\Big) \ll L(1, \operatorname{sym}^2 f) \ll (\log k)^2$$

we complete the proof (the lower bound in the above equation is proven in [7] while the upper bound is classical). □

We are now ready to prove Effective QUE.

Proof of Effective QUE. Let $\mathcal{R} = \{(x, y) : x \in I, y \in J\} \subset \mathcal{F}$ be a rectangular domain and write $\mathcal{R} = I \times J$ where I and J are intervals. Let $\mathcal{R}' = \mathcal{R} \cap \{z \in \mathcal{F} : \operatorname{Im}(z) \le (\log k)^{\eta_1}\}$, where $0 < \eta_1 \le 1$ will be chosen later, and note that $\mathcal{R}'$ is also a rectangular domain. We now will use a result of Soundararajan which bounds the amount of L^2-mass of $y^{k/2} f(z)$ high in the cusp. This enables us to restrict to rectangular regions of the form $\mathcal{R}'$. From the main result of Soundararajan [21] we have that

$$\iint\limits_{\substack{|\operatorname{Re}(z)| \le \frac{1}{2} \\ \operatorname{Im}(z) \ge (\log k)^{\eta_1}}} y^k |f(z)|^2 \frac{dx\, dy}{y^2} \ll \frac{1}{(\log k)^{\eta_1/2 - \varepsilon}}.$$

Thus,

$$\Big|\langle |F_k|^2, \chi_{\mathcal{R}}\rangle - \frac{3}{\pi} \operatorname{Area}_{\mathbb{H}}(\mathcal{R})\Big| = \Big|\langle |F_k|^2, \chi_{\mathcal{R}'}\rangle - \frac{3}{\pi} \operatorname{Area}_{\mathbb{H}}(\mathcal{R}')\Big| + O\Big(\frac{1}{(\log k)^{\eta_1/2 - \varepsilon}}\Big). \tag{4.15}$$

Hence, we may restrict our attention to rectangular domains lying inside $\{z \in \mathcal{F} : \operatorname{Im}(z) \le (\log k)^{\eta_1}\}$ at the cost of an error that is $O((\log k)^{-\eta_1/2+\varepsilon})$.

We now consider a smooth function $\phi_J(y)$ that approximates the characteristic function of the interval $J = [c, d]$, where $c \le C \le (\log k)^{\eta_1}$. Let $\phi_J(y)$ be such that $\phi_J(y) = 1$ for $y \in J$. Moreover, suppose that $\phi_J(y)$ is supported in $J_\delta = [c - \delta, d - \delta]$ and satisfies $\phi_J^{(\ell)}(y) \ll (1/\delta)^{\ell}$ for all $\ell \ge 1$. We also pick a $\phi_I(x)$ with identical properties. Consider $\phi(x, y) = \phi_I(x)\phi_J(x)$. Then we easily see that $\Delta^{\ell}\phi \ll (1/\delta)^{2\ell}$. We also choose $\delta = (\log k)^{-\eta_2}$, with $0 < \eta_2 \le 1$ to be chosen later.

According to Lemma 4.4 we have,

$$\begin{aligned}\langle |F_k|^2, \chi_{\mathcal{R}}\rangle =& \frac{3}{\pi}\cdot\langle 1, \chi_{\mathcal{R}}\rangle + O(\delta) + O\Big(C(\log k)^{\varepsilon}\prod_{p\le k}\Big(1 - \frac{(|\lambda_f(p)|-1)^2}{p}\Big)\Big) \\ &+ O(\delta^{-3}(\log k)^{-1+\varepsilon}).\end{aligned} \tag{4.16}$$

On the other hand, according to Lemma 4.3 we have

$$\begin{aligned}\langle |F_k|^2, \chi_{\mathcal{R}}\rangle =& \frac{3}{\pi}\cdot\langle 1, \chi_R\rangle + O(\delta) + O_A((\log k)^{-A}) \\ &+ O\Big((1/\delta)^{3/2}\cdot(\log k)^{\varepsilon}\Big(\prod_{p\le k}\Big(1-\frac{n(p)}{p}\Big) + \prod_{p\le k}\Big(1 - \frac{\lambda_f(p^2)+1}{p}\Big)\Big).\end{aligned} \tag{4.17}$$

We now balance the error terms with Euler products. First we set $\eta_1 = 3/2\cdot\eta_2$ so it remains to optimize

$$\min\Big(\prod_{p\le k}\Big(1-\frac{(|\lambda_f(p)|-1)^2}{p}\Big)\ ,\ \prod_{p\le k}\Big(1-\frac{n(p)}{p}\Big) + \prod_{p\le k}\Big(1-\frac{\lambda_f(p^2)+1}{p}\Big)\Big).$$

For $a, b, c \ge 0$ we have

$$\min(a, b+c) \le \min(a,b) + \min(a,c) \ll a^{\alpha}b^{1-\alpha} + a^{\beta}c^{1-\beta}.$$

Therefore it is enough to choose α and β so as to minimize separately $a^{\alpha}c^{1-\alpha}$ and $b^{\beta}c^{1-\beta}$ for a, b, c corresponding to the Euler products above. To shorten notation write $\lambda = |\lambda_f(p)|$. This leads us to finding an $0 \le \alpha \le 1$ which minimizes

$$\max_{0\le\lambda\le 2}\Big(-\alpha(\lambda-1)^2 - (1-\alpha)(\lambda^2 - 1 - \tfrac{1}{4}(\lambda^2-1)^2 + \tfrac{1}{4})\Big).$$

We also need to find a $0 \le \beta \le 1$ which will minimize

$$\max_{0\le\lambda\le 2}\Big(-\beta(\lambda-1)^2 - (1-\beta)\lambda^2\Big).$$

The second condition is easily optimized by taking $\beta = \frac{1}{2}$ and under this choice the maximum is less than $-\frac{1}{4}$. For the first condition we note that we can restrict to $\lambda \le 1$, because for $\lambda \ge 1$ the max is always bounded by $-\frac{1}{4}$. In the range $0 \le \lambda \le 1$, we have $\frac{1}{4}(\lambda^2-1)^2 \le (\lambda-1)^2$. Thus it's enough to optimize

$$\max_{0\le\lambda\le 1}\Big(-\alpha(\lambda-1)^2 - (1-\alpha)(\lambda^2 - 1 - \tfrac{1}{4}(\lambda-1)^2 + \tfrac{1}{4}\Big).$$

For $\frac{1}{5} \le \alpha \le 1$ this maximum is equal to

$$\frac{17\alpha}{4} + \frac{64}{\alpha+3} - \frac{81}{4}.$$

This is smallest when $\alpha = -3 + 16/\sqrt{17}$ and the minimum is then

$$-\kappa := 8\sqrt{17} - 33 = -0.01515499\ldots.$$

Thus, the error terms in (4.16) and (4.17) with Euler products are $\ll (\log k)^{\eta_1-\kappa+\varepsilon}$. Since we chose $\eta_1 = 3/2\eta_2$ the other error terms from (4.16), (4.17), and (4.15) are $\ll (\log k)^{\varepsilon}((\log k)^{-\eta_1/2} + (\log k)^{2\eta_1-1} + (\log k)^{-2/3\eta_1})$. So this balances by taking $\eta_1 = 2/3\kappa$ and gives an error of $O((\log k)^{-\kappa/3})$ as claimed. □

Acknowledgments. The third author is grateful to Zeev Rudnick for inviting him to Tel-Aviv University where this work was started. We would also like to thank Misha Sodin and Zeev Rudnick for many stimulating and helpful discussions. Additionally, we are also grateful to Roman Holowinsky for sending us lecture notes from Henryk Iwaniec's course on QUE.

References

1. Amit Ghosh and Peter Sarnak, *Real zeros of holomorphic Hecke cusp forms*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 2, 465–487. MR 2881302
2. R. R. Hall and G. Tenenbaum, *Effective mean value estimates for complex multiplicative functions*, Math. Proc. Cambridge Philos. Soc. **110** (1991), no. 2, 337–351. MR 1113432 (93e:11109)
3. Gergely Harcos, *Uniform approximate functional equation for principal L-functions*, Int. Math. Res. Not. (2002), no. 18, 923–932. MR 1902296 (2003d:11074)
4. D. R. Heath-Brown, *Convexity bounds for L-functions*, Acta Arith. **136** (2009), no. 4, 391–395. MR 2476604 (2010b:11115)
5. Jeffrey Hoffstein and Paul Lockhart, *Coefficients of Maass forms and the Siegel zero*, Ann. of Math. (2) **140** (1994), no. 1, 161–181, With an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman. MR 1289494 (95m:11048)
6. Roman Holowinsky, *Sieving for mass equidistribution*, Ann. of Math. (2) **172** (2010), no. 2, 1499–1516. MR 2680498 (2011i:11060)
7. Roman Holowinsky and Kannan Soundararajan, *Mass equidistribution for Hecke eigenforms*, Ann. of Math. (2) **172** (2010), no. 2, 1517–1528. MR 2680499 (2011i:11061)
8. Henryk Iwaniec and Emmanuel Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004. MR 2061214 (2005h:11005)
9. Henry H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. MR 1937203 (2003k:11083)
10. Y.-K. Lau and J. Wu, *A large sieve inequality of Elliott-Montgomery-Vaughan type for automorphic forms and two applications*, Int. Math. Res. Not. IMRN (2008), no. 5, Art. ID rnm 162, 35. MR 2418290 (2010b:11128)
11. B. Ja. Levin, *Distribution of zeros of entire functions*, revised ed., Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, R.I., 1980, Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman. MR 589888 (81k:30011)

12. Xiannan Li, *Upper bounds on L-functions at the edge of the critical strip*, Int. Math. Res. Not. IMRN (2010), no. 4, 727–755. MR 2595006 (2011a:11160)
13. Wenzhi Luo and Peter Sarnak, *Mass equidistribution for Hecke eigenforms*, Comm. Pure Appl. Math. **56** (2003), no. 7, 874–891, Dedicated to the memory of Jürgen K. Moser. MR 1990480 (2004e:11038)
14. ______, *Quantum variance for Hecke eigenforms*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 5, 769–799. MR 2103474 (2005k:11101)
15. Kaisa Matomäki, *Real zeros of holomorphic hecke cusp forms and sieving short intervals*, J. Eur. Math. Soc. **to appear** (2014), 22 pages.
16. Kaisa Matomäki and Maksym Radziwiłł, *Multiplicative functions in short intervals*, preprint. (2015).
17. Hugh L. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, CBMS Regional Conference Series in Mathematics, vol. 84, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1994. MR 1297543 (96i:11002)
18. M. Ram Murty and Kaneenika Sinha, *Effective equidistribution of eigenvalues of Hecke operators*, J. Number Theory **129** (2009), no. 3, 681–714. MR 2488597 (2010j:11071)
19. Zeév Rudnick, *On the asymptotic distribution of zeros of modular forms*, Int. Math. Res. Not. (2005), no. 34, 2059–2074. MR 2181743 (2006k:11099)
20. Atle Selberg, *On the normal density of primes in small intervals, and the difference between consecutive primes*, Arch. Math. Naturvid. **47** (1943), no. 6, 87–105. MR 0012624 (7,48e)
21. Kannan Soundararajan, *Quantum unique ergodicity for* $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, Ann. of Math. (2) **172** (2010), no. 2, 1529–1538. MR 2680500 (2011j:11098)
22. ______, *Weak subconvexity for central values of L-functions*, Ann. of Math. (2) **172** (2010), no. 2, 1469–1498. MR 2680497 (2011i:11077)
23. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, second ed., The Clarendon Press Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown. MR 882550 (88c:11049)
24. Thomas Watson, *Rankin triple products and quantum chaos*, available on the arXiv at **arXiv:0810.0425** (2008), 58 pages.
25. Honggang Xia, *On* L^∞ *norms of holomorphic cusp forms*, J. Number Theory **124** (2007), no. 2, 325–327. MR 2321365 (2008h:11036)
26. Matt Young, *The quantum unique ergodicity conjecture for thin sets*, available on the arXiv at **arXiv:1306.1554** (2013), 53 pages.

School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel
E-mail address: slester@post.tau.ac.il

Department of Mathematics and Statistics, University of Turku, 20014 Turku, Finland
E-mail address: ksmato@utu.fi

Centre de Recherches Mathematiques Universite de Montreal P. O Box 6128, Centre-Ville Station Montreal, Quebec H3C 3J7
Current address: Department of Mathematics, Rutgers University, Hill Center for the Mathematical Sciences, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019
E-mail address: radziwil@crm.umontreal.edu