AN AVERAGED FORM OF CHOWLA'S CONJECTURE

KAISA MATOMÄKI, MAKSYM RADZIWIŁŁ, AND TERENCE TAO

Abstract. Let λ denote the Liouville function. A well known conjecture of Chowla asserts that for any distinct natural numbers $h_1, \dots, h_k$, one has $\sum_{1 \le n \le X} \lambda(n + h_1) \cdots \lambda(n + h_k) = o(X)$ as $X \to \infty$. This conjecture remains unproven for any $h_1, \dots, h_k$ with $k \ge 2$. In this paper, using the recent results of the first two authors on mean values of multiplicative functions in short intervals, combined with an argument of Katai and Bourgain-Sarnak-Ziegler, we establish an averaged version of this conjecture, namely

$$\sum_{h_1, \dots, h_k \le H} \left| \sum_{1 \le n \le X} \lambda(n + h_1) \cdots \lambda(n + h_k) \right| = o(H^k X)$$

as $X \to \infty$ whenever $H = H(X) \le X$ goes to infinity as $X \to \infty$, and k is fixed. Related to this, we give the exponential sum estimate

$$\int_0^X \left| \sum_{x \le n \le x + H} \lambda(n) e(\alpha n) \right| dx = o(HX)$$

as $X \to \infty$ uniformly for all $\alpha \in \mathbb{R}$, with H as before. Our arguments in fact give quantitative bounds on the decay rate (roughly on the order of $\frac{\log \log H}{\log H}$), and extend to more general bounded multiplicative functions than the Liouville function, yielding an averaged form of a (corrected) conjecture of Elliott.

1. Introduction

Let $\lambda : \mathbb{N} \to \{-1, +1\}$ be the Liouville function, that is to say the completely multiplicative function such that $\lambda(p) = -1$ for all primes p. The prime number theorem implies that [1]

$$\sum_{1 \le n \le X} \lambda(n) = o(X)$$

as $X \to \infty$. More generally, a famous conjecture of Chowla [3] asserts that for any distinct natural numbers $h_1, \dots, h_k$, one has

$$\sum_{1 \le n \le X} \lambda(n + h_1) \cdots \lambda(n + h_k) = o(X) \tag{1.1}$$

as $X \to \infty$.

Chowla's conjecture remains open for any $h_1, \dots, h_k$ with $k \ge 2$. Our first main theorem establishes an averaged form of this conjecture:

[1] See Section 1.3 below for our asymptotic notation conventions.

Theorem 1.1 (Chowla's conjecture on average). *For any natural number k, and any $10 \leq H \leq X$, we have*

$$\sum_{1 \leq h_1, \ldots, h_k \leq H} \left| \sum_{1 \leq n \leq X} \lambda(n+h_1) \cdots \lambda(n+h_k) \right| \ll k \left(\frac{\log \log H}{\log H} + \frac{1}{\log^{1/3000} X} \right) H^k X. \quad (1.2)$$

In fact, we have the slightly stronger bound

$$\sum_{1 \leq h_2, \ldots, h_k \leq H} \left| \sum_{1 \leq n \leq X} \lambda(n)\lambda(n+h_2) \cdots \lambda(n+h_k) \right| \ll k \left(\frac{\log \log H}{\log H} + \frac{1}{\log^{1/3000} X} \right) H^{k-1} X. \quad (1.3)$$

In the case $k = 2$ our result implies that

$$\sum_{1 \leq h \leq H} \left| \sum_{1 \leq n \leq X} \lambda(n)\lambda(n+h) \right| = o(HX)$$

provided that $H \to \infty$ arbitrarily slowly with $X \to \infty$ (and $H \leq X$). Note that the $k = 2$ case of Chowla's conjecture is equivalent to the above asymptotic holding in the case that H is bounded rather than going to infinity.

In fact, we have a more precise bound than (1.2) (or (1.3)) that gives more control on the exceptional tuples $(h_1, \ldots, h_k)$ for which the sums $\sum_{1 \leq n \leq X} \lambda(n+h_1) \cdots \lambda(n+h_k)$ are large; see Remark 5.2 below. In particular in the special case $k = 2$ we get the following result.

Theorem 1.2. *Let $\delta \in (0, 1]$ be fixed. There is a large but fixed $H = H(\delta)$ such that, for all large enough X,*

$$\left| \sum_{1 \leq n \leq X} \lambda(n)\lambda(n+h) \right| \leq \delta X \quad (1.4)$$

for all but at most $H^{1-\frac{\delta}{5000}}$ integers $|h| \leq H$.

One can also replace the ranges $1 \leq h_j \leq H$ in Theorem 1.1 by $b_j + 1 \leq h_j \leq b_j + H$ for any $b_j = O(X)$; see Theorem 1.6 below.

The exponents $1/3000$ and $1/5000$ in the above theorems may certainly be improved, but we did not attempt to optimize the constants here. However, our methods cannot produce a gain much larger than $\frac{1}{\log H}$, as one would then have to somehow control λ on numbers that are not divisible by any prime less than H, at which point we are no longer able to exploit the averaging in the $h_1, \ldots, h_k$ parameters. It would be of particular interest to obtain a gain of more than $\frac{1}{\log X}$, as one could then potentially localize λ to primes and obtain some version of the prime tuples conjecture when the $h_1, \ldots, h_k$ parameters are averaged over short intervals, but this is well beyond the capability of our methods. (If instead one is allowed to average the $h_1, \ldots, h_k$ over long intervals (of scale comparable to X), one can obtain various averaged forms of the prime tuples conjecture and its relatives, by rather different methods to those used here; see [1], [15], [13], [12], [9].)

Theorem 1.1 is closely related to the following averaged short exponential sum estimate, which may be of independent interest.

Theorem 1.3 (Exponential sum estimate)**.** *For any* $10 \leq H \leq X$*, one has*

$$\sup_{\alpha \in \mathbb{R}} \int_0^X \left| \sum_{x \leq n \leq x+H} \lambda(n) e(\alpha n) \right| \, dx \ll \left(\frac{\log \log H}{\log H} + \frac{1}{\log^{1/700} X} \right) HX.$$

Actually, for technical reasons it is convenient to prove a sharper version of Theorem 1.3 in which the Liouville function has been restricted to those numbers that have "typical" factorization; see Theorem 2.3. This sharper version will then be used to establish Theorem 1.1.

The relationship between Theorem 1.1 and Theorem 1.3 stems from the following Fourier-analytic identity:

Lemma 1.4 (Fourier identity)**.** *If* $f : \mathbb{Z} \to \mathbb{C}$ *is a function supported on a finite set, and* $H > 0$*, then*

$$\int_{\mathbb{T}} \left(\int_{\mathbb{R}} \left| \sum_{x \leq n \leq x+H} f(n) e(\alpha n) \right|^2 \, dx \right)^2 \, d\alpha = \sum_{|h| \leq H} (H - |h|)^2 \left| \sum_n f(n) \overline{f}(n+h) \right|^2.$$

Proof. Using the Fourier identity $\int_{\mathbb{T}} e(n\alpha) \, d\alpha = 1_{n=0}$, we can expand the left-hand side as

$$\sum_{n,n',m,m'} f(n) \overline{f}(n') f(m) \overline{f}(m') 1_{n+m-n'-m'=0} \left(\int_{\mathbb{R}} 1_{x \leq n,n' \leq x+H} \, dx \right) \left(\int_{\mathbb{R}} 1_{y \leq m,m' \leq y+H} \, dy \right).$$

Writing $n' = n + h$, we see that both integrals are equal to $H - |h|$ if $|h| \leq H$, and vanish otherwise. The claim follows. $\square$

Theorem 1.3 may be compared with the classical estimate

$$\sup_{\alpha \in \mathbb{R}} \left| \sum_{1 \leq n \leq X} \lambda(n) e(\alpha n) \right| \ll_A X \log^{-A} X$$

of Davenport [5], valid for any $A > 0$. Indeed, one can view Theorem 1.3 as asserting that a weak form of Davenport's estimate holds on average in short intervals. It would be of interest to also obtain non-trivial bounds on the larger quantity

$$\int_0^X \sup_{\alpha \in \mathbb{R}} \left| \sum_{x \leq n \leq x+H} \lambda(n) e(\alpha n) \right| \, dx$$

but this appears difficult to establish with our methods.

As with other applications of the circle method, our proof of Theorem 1.3 splits into two cases, depending on whether the quantity α is on "major arc" or on "minor arc". In the "major arc" case we are able to use the recent results of the first two authors [14] on the average size of mean values of multiplicative functions on short intervals. Actually, in order to handle the presence of complex Dirichlet characters, we need to extend the results in [14] to complex-valued multiplicative functions rather than real-valued ones; this is accomplished in an appendix to this paper (Appendix A). In the "minor arc" case we use a variant of the arguments of Katai [11] and Bourgain-Sarnak-Ziegler [2] (see also the earlier works of Montgomery-Vaughan [16] and Daboussi-Delange [4]) to obtain the required cancellation. One innovation here is to rely on a combinatorial identity of Ramaré (also used in [14]) as a substitute for the Turan-Kubilius inequality, as this leads to superior quantitative estimates (particularly if one first restricts the variable n to have a "typical" prime factorization).

1.1. **Extension to more general multiplicative functions.** Define a 1-*bounded multiplicative function* to be a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ such that $|f(n)| \leq 1$ for all $n \in \mathbb{N}$. Given two 1-bounded multiplicative functions f, g and a parameter $X \geq 1$, we define the distance $\mathbb{D}(f, g; X) \in [0, +\infty)$ by the formula

$$\mathbb{D}(f, g; X) := \left(\sum_{p \leq X} \frac{1 - \mathrm{Re}(f(p)\overline{g(p)})}{p} \right)^{1/2}.$$

This is known to give a (pseudo-)metric on 1-bounded multiplicative functions; see [8, Lemma 3.1]. We also define the asymptotic counterpart $\mathbb{D}(f, g; \infty) \in [0, +\infty]$ by the formula

$$\mathbb{D}(f, g; \infty) := \left(\sum_{p} \frac{1 - \mathrm{Re}(f(p)\overline{g(p)})}{p} \right)^{1/2}.$$

We informally say that f *pretends* to be g if $\mathbb{D}(f, g; X)$ (or $\mathbb{D}(f, g; \infty)$) is small (or finite).

For any 1-bounded multiplicative function g and real number $X > 1$, we introduce the quantity

$$M(g; X) := \inf_{|t| \leq X} \mathbb{D}(g, n \mapsto n^{it}; X)^2, \tag{1.5}$$

and then the more general quantity

$$\begin{aligned} M(g; X, Q) &:= \inf_{q \leq Q; \chi\ (q)} M(g\overline{\chi}; X) \\ &= \inf_{|t| \leq X; q \leq Q; \chi\ (q)} \mathbb{D}(g, n \mapsto \chi(n) n^{it}; X)^2, \end{aligned}$$

where χ ranges over all Dirichlet characters of modulus $q \leq Q$. Informally, $M(g; X)$ is small when g pretends to be like a multiplicative character $n \mapsto n^{it}$, and $M(g; X, Q)$ is small when g pretends to be like a twisted Dirichlet character of modulus at most Q and twist of height at most X. We also define the asymptotic counterpart

$$M(g; \infty, \infty) = \inf_{\chi, t} \mathbb{D}(g, n \mapsto \chi(n) n^{it}; \infty)^2$$

where χ now ranges over all Dirichlet characters and t ranges over all real numbers.

In [6, Conjecture II], Elliott proposed the following more general form of Chowla's conjecture, which we phrase here in contrapositive form.

Conjecture 1.5 (Elliott's conjecture)**.** *Let $g_1, \dots, g_k : \mathbb{N} \to \mathbb{C}$ be 1-bounded multiplicative functions, and let $a_1, \dots, a_k, b_1, \dots, b_k$ be natural numbers such that any two of the $(a_1, b_1), \dots, (a_k, b_k)$ are linearly independent in $\mathbb{Q}^2$. Suppose that there is an index $1 \leq j_0 \leq k$ such that*

$$M(g_{j_0}; \infty, \infty) = \infty. \tag{1.6}$$

Then

$$\sum_{1 \leq n \leq X} \prod_{j=1}^{k} g_j(a_j n + b_j) = o(X) \tag{1.7}$$

as $X \to \infty$.

Informally, this conjecture asserts that for pairwise linearly independent $(a_1, b_1), \dots, (a_k, b_k)$ and any 1-bounded multiplicative $g_1, \dots, g_k$, one has the asymptotic (1.7) as $X \to \infty$, unless each of the g_j pretends to be a twisted Dirichlet character $n \mapsto \chi_j(n) n^{it_j}$. Note that some condition of this form is necessary, since if $g(n)$ is equal to $\chi(n) n^{-it}$ then $g(n)\overline{g(n+h)}$ will be biased to be positive for large n, if h is fixed and divisible by the modulus q of χ; one also expects some bias when h is not divisible by this modulus since the sums $\sum_{n \in \mathbb{Z}/q\mathbb{Z}} \chi(n)\overline{\chi(n+h)}$ do not vanish in general. From the prime number theorem in arithmetic progressions it follows that

$$M(\lambda; \infty, \infty) = \infty,$$

so Elliott's conjecture implies Chowla's conjecture (1.1).

When one allows the functions g_j to be complex-valued rather than real-valued, Elliott's conjecture turns out to be false on a technicality; one can choose 1-bounded multiplicative functions g_j which are arbitrarily close at various scales to a sequence of functions of the form $n \mapsto n^{it_m}$ (which allows one to violate (1.7)) without *globally* pretending to be n^{it} (or $\chi(n) n^{it}$) for any *fixed* t; we present this counterexample in Appendix B. However, this counterexample can be removed by replacing (1.6) with the stronger condition that

$$M(g_{j_0}; X, Q) \to \infty \tag{1.8}$$

as $X \to \infty$ for each fixed Q. In the real-valued case, (1.8) and (1.6) are equivalent by a triangle inequality argument of Granville and Soundararajan which we give in Appendix C.

As evidence for the corrected form of Conjecture 1.5 (in both the real-valued and complex-valued cases), we present the following averaged form of that conjecture:

Theorem 1.6 (Elliott's conjecture on average)**.** *Let* $10 \leq H \leq X$ *and* $A \geq 1$. *Let* $g_1, \dots, g_k : \mathbb{N} \to \mathbb{C}$ *be* 1*-bounded multiplicative functions, and let* $a_1, \dots, a_k, b_1, \dots, b_k$ *be natural numbers with* $a_j \leq A$ *and* $b_j \leq AX$ *for* $j = 1, \dots, k$. *Then for any* $1 \leq j_0 \leq k$ *one has*

$$\begin{aligned} &\sum_{1 \leq h_1, \dots, h_k \leq H} \left| \sum_{1 \leq n \leq X} \prod_{j=1}^k g_j(a_j n + b_j + h_j) \right| \\ &\qquad \ll A^2 k \left(\exp(-M/80) + \frac{\log\log H}{\log H} + \frac{1}{\log^{1/3000} X} \right) H^k X \end{aligned} \tag{1.9}$$

where

$$M := M(g_{j_0}; 10AX, Q)$$

and

$$Q := \min(\log^{1/125} X, \log^{20} H).$$

In fact, we have the slightly stronger bound

$$\begin{aligned} &\sum_{1 \leq h_2, \dots, h_k \leq H} \left| \sum_{1 \leq n \leq X} g_1(a_1 n + b_1) \prod_{j=2}^k g_j(a_j n + b_j + h_j) \right| \\ &\qquad \ll A^2 k \left(\exp(-M/80) + \frac{\log\log H}{\log H} + \frac{1}{\log^{1/3000} X} \right) H^{k-1} X. \end{aligned} \tag{1.10}$$

Note that if $a_1, \dots, a_k, b_1, \dots, b_k$ are fixed, g_{j_0} is independent of X and obeys the condition (1.8) for any fixed Q, and $H = H(X)$ is chosen to go to infinity arbitrarily slowly as $X \to \infty$, then the quantity M in the above theorem goes to infinity (note that $M(g; X, Q)$ is non-decreasing in Q), and (1.10) then implies an averaged form of the asymptotic (1.7). Thus Theorem 1.6 is indeed an averaged form of the corrected form of Conjecture 1.5. (We discovered the counterexample in Appendix B while trying to interpret Theorem 1.6 as an averaged version of the original form of Conjecture 1.5.)

For $g(n) = \lambda(n)$ and X, Q, M as in the above theorem, one obtains, for every $\varepsilon > 0$, the bound

$$M \geq \inf_{|t| \leq X; q \leq Q; \chi\ (q)} \sum_{\exp((\log X)^{2/3+\varepsilon}) \leq p \leq X} \frac{1 + \operatorname{Re} \chi(p) p^{it}}{p} \geq \left(\frac{1}{3} - \varepsilon\right) \log\log X + O(1) \tag{1.11}$$

where the last inequality is established via standard methods from the Vinogradov-Korobov type zero-free region

$$\left\{ \sigma + it : \sigma > 1 - \frac{c}{\max\{\log q, (\log(3+|t|))^{2/3} (\log\log(3+|t|))^{1/3}\}} \right\}$$

for $L(s, \chi)$ and some absolute constant $c > 0$, which applies since χ has conductor $q \leq (\log X)^{1/125}$ (so that there are no exceptional zeros), see [17, §9.5]. Hence Theorem 1.6 implies Theorem 1.1. The same argument gives Theorem 1.1 when the Liouville function λ is replaced by the Möbius function μ.

We also have a generalized form of Theorem 1.3:

Theorem 1.7 (Exponential sum estimate). *Let $X \geq H \geq 10$ and let g be a 1-bounded multiplicative function. Then*

$$\sup_{\alpha \in \mathbb{T}} \int_0^X \Big| \sum_{x \leq n \leq x+H} g(n) e(\alpha n) \Big|\, dx \ll \left(\exp(-M(g; X, Q)/20) + \frac{\log\log H}{\log H} + \frac{1}{\log^{1/700} X} \right) HX$$

where

$$Q := \min(\log^{1/125} X, \log^5 H).$$

By (1.11), Theorem 1.7 implies Theorem 1.3.

1.2. **Acknowledgments.** TT was supported by a Simons Investigator grant, the James and Carol Collins Chair, the Mathematical Analysis & Application Research Fund Endowment, and by NSF grant DMS-1266164. The authors thank Andrew Granville for useful comments and corrections.

1.3. **Notation.** Our asymptotic notation conventions are as follows. We use $X \ll Y$, $Y \gg X$, or $X = O(Y)$ to denote the estimate $|X| \leq CY$ for some absolute constant C. If x is a parameter going to infinity, we use $X = o(Y)$ to denote the claim that $|X| \leq c(x) Y$ for some quantity $c(x)$ that goes to zero as $x \to \infty$ (holding all other parameters fixed).

Unless otherwise specified, all sums are over the integers, except for sums over the variable p (or p_1, p_2, etc.) which are understood to be over primes.

We use $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ to denote the standard unit circle, and let $e : \mathbb{T} \to \mathbb{C}$ be the standard character $e(x) := e^{2\pi i x}$.

We use 1_S to denote the indicator of a predicate S, thus $1_S = 1$ when S is true and $1_S = 0$ when S is false. If A is a set, we write $1_A(n)$ for $1_{n\in A}$, so that 1_A is the indicator function of A.

2. Restricting to numbers with typical factorization

To prove Theorem 1.6 and Theorem 1.7 (and hence Theorem 1.1 and Theorem 1.3), it is technically convenient (as in the previous paper [14] of the first two authors) to restrict the support of the multiplicative functions to a certain dense set $\mathcal{S}$ of natural numbers that have a "typical" prime factorization in a certain specific sense, in order to fully exploit a useful combinatorial identity of Ramaré (see (3.2) below). This will lead to improved quantitative estimates in the arguments in subsequent sections of the paper.

More precisely, we introduce the following sets $\mathcal{S}$ of numbers with typical prime factorization, which previously appeared in [14].

Definition 2.1. Let $10 < P_1 < Q_1 \le X$ and $\sqrt{X} \le X_0 \le X$ be quantities such that $Q_1 \le \exp(\sqrt{\log X_0})$. We then define P_j, Q_j for $j > 1$ by the formula

$$P_j := \exp(j^{4j}(\log Q_1)^{j-1}\log P_1); \quad Q_j := \exp(j^{4j+2}(\log Q_1)^j).$$

for $j > 1$; note that the intervals $[P_j, Q_j]$ are disjoint and increase to infinity. Let J be the largest index such that $Q_J \le \exp(\sqrt{\log X_0})$. Then we define $\mathcal{S}_{P_1,Q_1,X_0,X}$ to be the set of all the numbers $1 \le n \le X$ which have at least one prime factor in the interval $[P_j, Q_j]$ for each $1 \le j \le J$.

This set is fairly dense if P_1 and Q_1 are widely separated:

Lemma 2.2. *Let $10 < P_1 < Q_1 \le X$ and $\sqrt{X} \le X_0 \le X$ be such that $Q_1 \le \exp(\sqrt{\log X_0})$. Then, for every large enough X,*

$$\#\{1 \le n \le X : n \notin \mathcal{S}_{P_1,Q_1,X_0,X}\} \ll \frac{\log P_1}{\log Q_1} \cdot X.$$

Proof. From the fundamental lemma of sieve theory (see e.g. [7, Theorem 6.17]) we know that, for any $1 \le j \le J$ and large enough X, the number of $1 \le n \le X$ that are not divisible by any prime in $[P_j, Q_j]$ is at most

$$\ll X \prod_{P_j \le p \le Q_j} \left(1 - \frac{1}{p}\right) \ll \frac{\log P_j}{\log Q_j} X = \frac{1}{j^2} \frac{\log P_1}{\log Q_1} X.$$

Summing over j, we obtain the claim. □

Both Theorem 1.6 and Theorem 1.7 will be deduced from the following claim.

Theorem 2.3 (Key exponential sum estimate)**.** *Let $X, H, W \ge 10$ be such that*

$$(\log H)^5 \le W \le \min\{H^{1/250}, (\log X)^{1/125}\}$$

and let g be a 1-bounded multiplicative function such that

$$W \le \exp(M(g; X, W)/3). \tag{2.1}$$

Set

$$\mathcal{S} := \mathcal{S}_{P_1,Q_1,\sqrt{X},X}$$

where

$$P_1 := W^{200}; \quad Q_1 := H/W^3.$$

Then for any $\alpha \in \mathbb{T}$, one has

$$\int_{\mathbb{R}} \left| \sum_{x \le n \le x+H} 1_{\mathcal{S}} g(n) e(\alpha n) \right| \, dx \ll \frac{(\log H)^{1/4} \log\log H}{W^{1/4}} HX. \tag{2.2}$$

In Section 5 we will show how this theorem implies Theorem 1.6. For now, let us at least see how it implies Theorem 1.7:

Proof. (Proof of Theorem 1.7 assuming Theorem 2.3) We may assume that X, H, and $M(g; X, Q)$ are larger than any specified absolute constant, as the claim is trivial otherwise.

Choose H_0 such that

$$\log H_0 := \min \left(\log^{1/700} X \log\log X, \exp(M(g; X, Q)/20) M(g; X, Q) \right).$$

We divide into two cases: $H \le H_0$ and $H > H_0$.

First suppose that $H \le H_0$. Then if we set $W := \log^5 H$, one verifies that all the hypotheses of Theorem 2.3 hold, and hence

$$\int_0^X \left| \sum_{x \le n \le x+H} 1_{\mathcal{S}} g(n) e(\alpha n) \right| \, dx \ll \frac{\log\log H}{\log H} HX.$$

On the other hand, from Lemma 2.2, the choice of W, P_1, Q_1, and the bound on H we see that

$$\#\{1 \le n \le X + H : n \notin \mathcal{S}\} \ll \frac{\log\log H}{\log H} X$$

and thus by Fubini's theorem and the triangle inequality

$$\int_0^X \left| \sum_{x \le n \le x+H} (1 - 1_{\mathcal{S}}) g(n) e(\alpha n) \right| \, dx \ll \frac{\log\log H}{\log H} HX.$$

Summing, we obtain Theorem 1.7 in this case.

Now suppose that $H > H_0$. Covering $[0, H]$ by $O(H/H_0)$ intervals of length H_0, we see that

$$\int_0^X \left| \sum_{x \le n \le x+H} g(n) e(\alpha n) \right| \, dx \ll \frac{H}{H_0} \int_0^{X+H} \left| \sum_{x \le n \le x+H_0} g(n) e(\alpha n) \right| \, dx.$$

Also, observe from the choice of H_0 that the quantity $\exp(-M(g; X, Q)/20) + \frac{\log\log H}{\log H} + \frac{1}{\log^{1/700} X}$ is unchanged up to multiplicative constans if one reduces H to H_0. Finally, from Mertens' theorem we see that $M(g; X + H, Q) = M(g; X, Q) + O(1)$. The claim then follows from the $H = H_0$ case (after performing the minor alteration of replacing X with $X + H$). □

We now begin the proof of Theorem 2.3. The first step is to reduce to the case where g is completely multiplicative rather than multiplicative. More precisely, we will deduce Theorem 2.3 from

Proposition 2.4 (Completely multiplicative exponential sum estimate). *Let $X, H, W \geq 10$ be such that*

$$(\log H)^5 \leq W \leq \min\{H^{1/250}, (\log X)^{1/125}\},$$

and let g be a 1-bounded completely multiplicative function such that

$$W \leq \exp(M(g; X, W)/3). \tag{2.3}$$

Let d be a natural number with $d < W$. Set

$$\mathcal{S} := \mathcal{S}_{P_1, Q_1, \sqrt{X}, X/d}$$

where

$$P_1 := W^{200}; \quad Q_1 := H/W^3.$$

Then for any $\alpha \in \mathbb{T}$ one has

$$\int_{\mathbb{R}} \left| \sum_{x/d \leq n \leq x/d + H/d} 1_{\mathcal{S}} g(n) e(\alpha n) \right| dx \ll \frac{1}{d^{3/4}} \frac{(\log H)^{1/4} \log\log H}{W^{1/4}} HX. \tag{2.4}$$

Let us explain why Theorem 2.3 follows from Proposition 2.4. Let the hypotheses and notation be as in Theorem 2.3. The function g is not necessarily completely multiplicative, but we may approximate it by the 1-bounded completely multiplicative function $g_1 : \mathbb{N} \to \mathbb{C}$, defined as the completely multiplicative function with $g_1(p) = g(p)$ for all primes p. By Möbius inversion we may then write $g = g_1 * h$ where $*$ denotes Dirichlet convolution and h is the multiplicative function $h = g * \mu g_1$. Observe that for all primes p, $h(p) = 0$ and $|h(p^j)| \leq 2$ for $j \geq 2$. We now write

$$\sum_{x \leq n \leq x+H} 1_{\mathcal{S}_{P_1, Q_1, \sqrt{X}, X}} g(n) e(\alpha n) = \sum_{d=1}^{\infty} h(d) \sum_{x/d \leq m \leq x/d + H/d} 1_{\mathcal{S}_{P_1, Q_1, \sqrt{X}, X}}(dm) g_1(m) e(d\alpha m)$$

and so by the triangle inequality we may upper bound the left-hand side of (2.2) by

$$\sum_{d=1}^{\infty} |h(d)| \int_{\mathbb{R}} \left| \sum_{x/d \leq m \leq x/d + H/d} 1_{\mathcal{S}_{P_1, Q_1, \sqrt{X}, X}}(dm) g_1(m) e(d\alpha m) \right| dx.$$

Let us first dispose of the contribution where $d \geq W$. Here we trivially bound this contribution by

$$\sum_{d \geq W} |h(d)| \sum_{m \leq (2X+H)/d} O(H)$$

(after moving the absolute values inside the m summation and then performing the integration on x first). We can bound this in turn by

$$\ll HX \frac{1}{W^{1/4}} \sum_{d=1}^{\infty} \frac{|h(d)|}{d^{3/4}}.$$

From Euler products we see that $\sum_{d=1}^{\infty} \frac{|h(d)|}{d^{3/4}} = O(1)$, so the contribution of this case is acceptable.

Now we consider the contribution $d < W < P_1$. In this case we may reduce

$$1_{\mathcal{S}_{P_1, Q_1, \sqrt{X}, X}}(dm) = 1_{\mathcal{S}_{P_1, Q_1, \sqrt{X}, X/d}}(m)$$

and so this contribution to (2.2) can be upper bounded by

$$\sum_{1\le d<W} |h(d)| \int_{\mathbb{R}} \left| \sum_{x/d\le m\le x/d+H/d} 1_{\mathcal{S}_{P_1,Q_1,\sqrt{X},X/d}}(m) g_1(m) e(d\alpha m)\right| \ dx.$$

By Proposition 2.4, this is bounded by

$$\sum_{d=1}^{\infty} \frac{|h(d)|}{d^{3/4}} \frac{(\log H)^{1/4}\log\log H}{W^{1/4}} HX.$$

As before we have $\sum_{d=1}^{\infty} \frac{|h(d)|}{d^{3/4}} = O(1)$, and Theorem 2.3 follows.

It remains to prove Proposition 2.4. For any $\alpha \in \mathbb{T}$, we know from the Dirichlet approximation theorem that there exists a rational number $\frac{a}{q}$ with $(a,q)=1$ and $1 \le q \le H/W$ such that

$$\left|\alpha - \frac{a}{q}\right| \le \frac{W}{qH} \le \frac{1}{q^2}.$$

In the next two sections, we will apply separate arguments to prove Proposition 2.4 in the *minor arc* case $q > W$ and the *major arc* case $q \le W$.

3. Proof of minor arc estimate

We now prove Proposition 2.4 in the minor arc case $q > W$. It suffices to show that

$$\int_{\mathbb{R}} \theta(x) \sum_{x/d\le n\le x/d+H/d} 1_{\mathcal{S}} g(n) e(\alpha n) \ dx \ll \frac{1}{d^{3/4}} \frac{(\log H)^{1/4}\log\log H}{W^{1/4}} HX \tag{3.1}$$

whenever $\theta : \mathbb{R} \to \mathbb{C}$ is measurable with $|\theta(x)| \le 1$ for all x and supported on $[0, X]$. We will now use a variant of an idea of Bourgain-Sarnak-Ziegler [2] (building on earlier works of Katai [11], Montgomery-Vaughan [16] and Daboussi-Delange [4]).

Let $\mathcal{P}$ be the set consisting of the primes lying between P_1 and Q_1. Then, notice that each $n \in \mathcal{S}$ has at least one prime factor from $\mathcal{P}$. This leads to the following variant of Ramaré's identity (see [7, Section 17.3]):

$$1_{\mathcal{S}}(n) = \sum_{p\in\mathcal{P}, m: mp=n} \frac{1_{\mathcal{S}'}(mp)}{1 + \#\{q \mid m : q \in \mathcal{P}\}}, \tag{3.2}$$

where $\mathcal{S}'$ is the set of all $1 \le n \le X/d$ that have at least one prime factor in each of the intervals $[P_j, Q_j]$ for $j \ge 2$.

Using this identity, we may write the left-hand side of (3.1) as

$$\sum_{p\in\mathcal{P}} \sum_{m} \frac{1_{\mathcal{S}'}(mp) g(mp) e(mp\alpha)}{1 + \#\{q|m : q \in \mathcal{P}\}} \int_{\mathbb{R}} \theta(x) 1_{x/d\le mp\le (x+H)/d} \ dx.$$

As g is completely multiplicative, $g(mp) = g(m)g(p)$. Thus it suffices to show that

$$\sum_{p\in\mathcal{P}} \sum_{m} \frac{1_{\mathcal{S}'}(mp) g(m) g(p) e(mp\alpha)}{1 + \#\{q|m : q \in \mathcal{P}\}} \int_{\mathbb{R}} \theta(x) 1_{x/d\le mp\le (x+H)/d} \ dx \ll$$

$$\ll \frac{(\log H)^{1/4}\log\log H}{d^{3/4}W^{1/4}} HX.$$

We can cover $\mathcal{P}$ by intervals $[P, 2P]$ with $P_1 \ll P \ll Q_1$ and P a power of two, and observe that

$$\sum_{\substack{P_1 \ll P \ll Q_1 \\ P = 2^j}} \frac{1}{\log P} \ll \log\log Q_1 - \log\log P_1 \ll \log\log H,$$

so by the triangle inequality it suffices to show that

$$\sum_{p \in \mathcal{P}: P \leq p \leq 2P} \sum_m \frac{1_{\mathcal{S}'}(mp) g(m) g(p) e(mp\alpha)}{1 + \#\{q | m : q \in \mathcal{P}\}} \int_{\mathbb{R}} \theta(x) 1_{x/d \leq mp \leq (x+H)/d}\, dx \ll \frac{(\log H)^{1/4}}{d^{3/4} W^{1/4} \log P} HX$$

for each such P. We can rearrange the left-hand side as

$$\sum_{m \in \mathcal{S}'} \frac{g(m)}{1 + \#\{q | m : q \in \mathcal{P}\}} \sum_{p \in \mathcal{P}: P \leq p \leq 2P} 1_{mp \leq X/d} g(p) e(mp\alpha) \int_{\mathbb{R}} \theta(x) 1_{x/d \leq mp \leq (x+H)/d}\, dx.$$

Observe that the summand vanishes unless $m \leq \frac{X}{dP}$. Crudely bounding[2] $\frac{g(m)}{1+\#\{q|m:q\in\mathcal{P}\}}$ in magnitude by 1 and applying Hölder's inequality, we may bound the previous expression in magnitude by

$$\left(\frac{X}{dP}\right)^{3/4} \left(\sum_{m \leq X/dP} \left| \sum_{p \in \mathcal{P}: P \leq p \leq 2P} 1_{mp \leq X/d} g(p) e(mp\alpha) \int_{\mathbb{R}} \theta(x) 1_{x/d \leq mp \leq (x+H)/d} \right|^4 dx \right)^{1/4}.$$

It thus suffices to show that

$$\sum_{m \leq X/dP} \left| \sum_{p \in \mathcal{P}: P \leq p \leq 2P} 1_{mp \leq X/d} g(p) e(mp\alpha) \int_{\mathbb{R}} \theta(x) 1_{x/d \leq mp \leq (x+H)/d} dx \right|^4 \ll \frac{\log H}{W \log^4 P} H^4 X P^3.$$

The left-hand side may be expanded as

$$\sum_{p_1, p_2, p_3, p_4 \in \mathcal{P}: P \leq p_1, p_2, p_3, p_4 \leq 2P} \int \cdots \int g(p_1) g(p_2) \overline{g(p_3) g(p_4)} \theta(x_1) \theta(x_2) \overline{\theta(x_3) \theta(x_4)}$$
$$\cdot \sum_{m \leq X/(dp_i), x_i/(dp_i) \leq m \leq (x_i+H)/(dp_i) \forall i=1,2,3,4} e(m(p_1 + p_2 - p_3 - p_4)\alpha) dx_1 dx_2 dx_3 dx_4.$$

From summing the geometric series, the summation over m is $O(\min(\frac{H}{P}, \frac{1}{\|(p_1+p_2-p_3-p_4)\alpha\|}))$, where $\|z\|$ denotes the distance from z to the nearest integer. Also, the sum vanishes unless we have $x_1 = O(X)$ and $x_i = x_1 p_i / p_1 + O(H)$ for $i = 2, 3, 4$, so there are only $O(XH^3)$ quadruples (x_1, x_2, x_3, x_4) which contribute here. Thus we may bound the previous expression by

$$O\left(XH^3 \sum_{p_1, p_2, p_3, p_4 \leq 2P} \min\left(\frac{H}{P}, \frac{1}{\|(p_1 + p_2 - p_3 - p_4)\alpha\|} \right) \right)$$

and so we reduce to showing that

$$\sum_{p_1, p_2, p_3, p_4 \leq 2P} \min\left(\frac{H}{P}, \frac{1}{\|(p_1 + p_2 - p_3 - p_4)\alpha\|} \right) \ll \log H \frac{HP^3}{W \log^4 P}. \tag{3.3}$$

[2]By using the Turan-Kubilius inequality here one could save a factor of $\log\log H$, but such a gain will not make a significant impact on our final estimates.

The quantity $p_1 + p_2 - p_3 - p_4$ is clearly of size $O(P)$. Conversely, from a standard upper bound sieve[3], the number of representations of an integer $n = O(P)$ of the form $p_1 + p_2 - p_3 - p_4$ with $p_1, p_2, p_3, p_4 \leq 2P$ prime is $O(\frac{P^3}{\log^4 P})$. Thus it suffices to show that

$$\sum_{n=O(P)} \min\left(\frac{H}{P}, \frac{1}{\|n\alpha\|}\right) \ll \frac{\log H}{W} H.$$

But from the Vinogradov lemma (see e.g. [10, Page 346]), the left-hand side is bounded by

$$O\left(\left(\frac{P}{q}+1\right)\left(\frac{H}{P} + q\log q\right)\right) \ll \frac{H}{q} + P\log q + \frac{H}{P} + q\log q$$

which, since

$$W^{200} = P_1 \ll P \ll Q_1 = H/W^3$$

and

$$W \leq q \leq H/W,$$

is bounded by $O(\frac{\log H}{W} H)$ as required.

4. Proof of major arc estimate

We now prove Proposition 2.4 in the major arc case $q \leq W$. We will discard the factor $(\log H)^{1/4} \log\log H$ on the right-hand side and show that

$$\int_{\mathbb{R}} \left| \sum_{x/d \leq n \leq (x+H)/d} 1_{\mathcal{S}} g(n) e(\alpha n) \right| \, dx \ll \frac{1}{d^{3/4} W^{1/4}} HX.$$

By hypothesis, we have $\alpha = \frac{a}{q} + \theta$ for some $\theta = O(\frac{W}{Hq})$. From the fundamental theorem of calculus we have

$$e(\alpha n) = e(an/q)\left(e(\theta(x/d + H/d)) - 2\pi i \theta \int_0^{H/d} 1_{n \leq x/d + H'} e(\theta(x/d + H'))\, dH'\right)$$

for $x/d \leq n \leq (x+H)/d$, and thus by the triangle inequality

$$\sum_{x/d \leq n \leq (x+H)/d} 1_{\mathcal{S}} g(n) e(\alpha n) \ll \left| \sum_{x/d \leq n \leq (x+H)/d} 1_{\mathcal{S}} g(n) e(an/q) \right| + \frac{W}{Hq} \int_0^{H/d} \left| \sum_{x/d \leq n \leq x/d + H'} 1_{\mathcal{S}} g(n) e(an/q) \right| \, dH'$$

for any x. By the triangle inequality again (and the hypothesis $q \leq W$), it thus suffices to show that

$$\int_{\mathbb{R}} \left| \sum_{x/d \leq n \leq x/d + H'} 1_{\mathcal{S}} g(n) e(an/q) \right| \, dx \ll \frac{q}{W} \frac{1}{d^{3/4} W^{1/4}} HX$$

[3]For instance, from [16, Theorem 3.13] one sees that any number $N = O(P)$ has $O(\frac{N}{\phi(N)} \frac{P}{\log^2 P})$ representations as the sum of two primes; since $\sum_{N=O(P)} \frac{N^2}{\phi(N)^2} = O(P)$ (see e.g. [16, Exercise 2.1.14]), the claim then follows from the Cauchy-Schwarz inequality.

for any $0 < H' \leq H/d$. Since $d < W$, this is trivial when $H' \leq H/W^2$, so we can assume that $H/W^2 \leq H' \leq H/d$. Splitting into residue classes modulo q and using the triangle inequality again, it suffices to show that

$$\int_{\mathbb{R}} \left| \sum_{x/d \leq n \leq x/d + H' : n = b\ (q)} 1_{\mathcal{S}} g(n) \right|\ dx \ll \frac{1}{W^{5/4} d^{3/4}} HX$$

for any residue class $b\ (q)$ (not necessarily primitive). Writing $b = d_0 b_0$, $q = d_0 q_0$ with $d_0 := (b, q)$, and then writing $n = d_0 m$, this becomes

$$\int_{\mathbb{R}} \left| \sum_{x/dd_0 \leq m \leq x/dd_0 + H'/d_0 : m = b_0\ (q_0)} 1_{\mathcal{S}} g(d_0 m) \right|\ dx \ll \frac{1}{W^{5/4} d^{3/4}} HX$$

Since g is completely multiplicative and $d_0 \leq q \leq W \leq P_1$, we have

$$1_{\mathcal{S}} g(d_0 m) = g(d_0) 1_{\mathcal{S}_{P_1, Q_1, \sqrt{X}, X/dd_0}} g(m)$$

and so we reduce to showing that

$$\int_{\mathbb{R}} \left| \sum_{x/dd_0 \leq m \leq x/dd_0 + H'/d_0 : m = b_0\ (q_0)} 1_{\mathcal{S}_{P_1, Q_1, \sqrt{X}, X/dd_0}} g(m) \right|\ dx \ll \frac{1}{W^{5/4} d^{3/4}} HX$$

Writing $d_1 := dd_0 \leq W^2$ and $H_1 := H'/d_0 \leq H/d_1 \leq H/d_1^{3/4}$, it thus suffices to show that

$$\int_{\mathbb{R}} \left| \sum_{x/d_1 \leq m \leq x/d_1 + H_1 : m = b_0\ (q_0)} 1_{\mathcal{S}_{P_1, Q_1, \sqrt{X}, X/d_1}} g(m) \right|\ dx \ll \frac{1}{W^{5/4}} H_1 X$$

for $H/W^3 \leq H_1 \leq H/d$.

The residue class $b_0\ (q_0)$ is primitive, and so by Fourier inversion we have

$$1_{b_0\ (q_0)}(m) = \frac{1}{\varphi(q_0)} \sum_{\chi\ (q_0)} \chi(b_0) \overline{\chi}(m)$$

where χ ranges over the Dirichlet characters of modulus q_0. By the triangle inequality, it thus suffices to show that

$$\int_{\mathbb{R}} \left| \sum_{x/d_1 \leq m \leq x/d_1 + H_1} 1_{\mathcal{S}_{P_1, Q_1, \sqrt{X}, X/d_1}} g\overline{\chi}(m) \right|\ dx \ll \frac{1}{W^{5/4}} H_1 X$$

for any Dirichlet character χ of modulus at most W. Making the change of variables $y = x/d_1$, we reduce to showing that

$$\int_{\mathbb{R}} \left| \sum_{y \leq m \leq y + H_1} 1_{\mathcal{S}_{P_1, Q_1, \sqrt{X}, X/d_1}} g\overline{\chi}(m) \right|\ dy \ll \frac{1}{W^{5/4}} H_1 \frac{X}{d_1}.$$

The summand vanishes unless $y \leq X/d_1$. The contribution of those y with $y \leq X/W^{10}$ is easily seen to be acceptable, so by dyadic decomposition it suffices to show that

$$\int_{X'}^{2X'} \left| \sum_{y \leq m \leq y + H_1} 1_{\mathcal{S}_{P_1, Q_1, \sqrt{X}, X/d_1}} g\overline{\chi}(m) \right|\ dy \ll \frac{1}{W^{5/4}} H_1 X'$$

for all $X/W^{10} \leq X' \leq X/d_1$ and $Q_1 = X/W^3 \leq H_1 \leq H$.

At this point we apply Theorem A.2 with $\eta = 1/20$ (note that $P_1 \geq (\log Q_1)^{40/\eta}$) to conclude that

$$\int_{X'}^{2X'} \left| \sum_{y \leq m \leq y+H_1} 1_{\mathcal{S}_{P_1,Q_1,\sqrt{X},X/d_1}} g\overline{\chi}(m) \right|^2 dy$$
$$\ll \left(\exp(-M(g\overline{\chi}; X'))M(g\overline{\chi}; X') + \frac{(\log H)^{1/3}}{P_1^{1/6-1/20}} + \frac{1}{(\log X')^{1/50}} \right) H_1^2 X'.$$

Since $P_1 = W^{200}$ and $W \geq \log^5 H$, we have

$$\frac{(\log H)^{1/3}}{P_1^{1/6-1/20}} \ll \frac{1}{W^{5/2}}$$

and certainly

$$\frac{1}{(\log X')^{1/50}} \ll \frac{1}{(\log X)^{1/50}} \ll \frac{1}{W^{5/2}}.$$

From Mertens' theorem and definition of $M(g, X, W)$,

$$M(g\overline{\chi}; X') \geq M(g\overline{\chi}; X) - O(1) \geq M(g, X, W) - O(1)$$

and thus by (2.3)

$$\exp(-M(g\overline{\chi}; X'))M(g\overline{\chi}; X') \ll \frac{1}{W^{5/2}}.$$

Putting all this together, we obtain

$$\int_{X'}^{2X'} \left| \sum_{y \leq m \leq y+H_1} 1_{\mathcal{S}_{P_1,Q_1,\sqrt{X},X/d_1}} g\overline{\chi}(m) \right|^2 dy \ll \frac{1}{W^{5/2}} H_1^2 X'$$

and the claim now follows from the Cauchy-Schwarz inequality.

5. Elliott's conjecture on the average

In this section we use Theorem 2.3 to prove Theorem 1.6. Theorem 1.6 will be deduced from the following result (compare also with Theorem 2.3 and deduction of Theorem 1.7 from it).

Proposition 5.1 (Truncated Elliott on the average)**.** *Let $X, H, W, A \geq 10$ be such that*

$$\log^{20} H \leq W \leq \min\{H^{1/500}, (\log X)^{1/125}\}.$$

Let $g_1, \ldots, g_k : \mathbb{N} \to \mathbb{C}$ be 1-bounded multiplicative functions, and let $a_1, \ldots, a_k, b_1, \ldots, b_k$ be natural numbers with $a_j \leq A$ and $b_j \leq 3AX$ for $j = 1, \ldots, k$. Let $1 \leq j_0 \leq k$ be such that

$$W \leq \exp(M(g_{j_0}; X, W)/3).$$

Set

$$\mathcal{S} = \mathcal{S}_{P_1,Q_1,\sqrt{10AX},10AX}$$

where

$$P_1 := W^{200}; \quad Q_1 := H^{1/2}/W^3.$$

Then

$$\sum_{1\le h_2,\dots,h_k\le H}\Big|\sum_{1\le n\le X} 1_{\mathcal S} g_1(a_1n+b_1)\prod_{j=2}^k 1_{\mathcal S} g_j(a_jn+b_j+h_j)\Big| \ll \frac{kA^2}{W^{1/20}}H^{k-1}X. \quad (5.1)$$

Proof of Theorem 1.6 assuming Proposition 5.1. We may assume that X, H, and M are larger than any specified absolute constant as the claim is trivial otherwise. We first make some initial reductions. The first estimate (1.9) of Theorem 1.6 follows from the second (1.10) after shifting b_1 by h_1 in (1.10) and averaging, provided that we relax the hypotheses $b_j \le AX$ slightly to $b_j \le 2AX$. Thus it suffices to prove (1.10) under the relaxed hypotheses $b_j \le 2AX$.

Let H_0 be such that

$$\log H_0 = \min\{\log^{1/3000} X \log\log X, \exp(M(g_{j_0}; 10AX, Q)/80)M(g_{j_0}; 10AX, Q)\}. \quad (5.2)$$

If $H \le H_0$ we take $W = \log^{20} H$ and let $\mathcal S$ be as in Proposition 5.1. All the assumptions of Proposition 5.1 hold and thus

$$\sum_{1\le h_2,\dots,h_k\le H}\Big|\sum_{1\le n\le X} 1_{\mathcal S} g_1(a_1n+b_1)\prod_{j=2}^k 1_{\mathcal S} g_j(a_jn+b_j+h_j)\Big| \ll \frac{kA^2}{\log H}H^{k-1}X.$$

Furthermore, from Lemma 2.2 we have

$$\sum_{n\le 10AX: n\notin\mathcal S} 1 \ll AX\frac{\log W}{\log H}. \quad (5.3)$$

From this and the triangle inequality, we have

$$\begin{aligned}\sum_{1\le n\le X} g_1(a_1n+b_1)&\prod_{j=2}^k g_j(a_jn+b_j+h_j)\\ &= \sum_{1\le n\le X} 1_{\mathcal S} g_1(a_1n+b_1)\prod_{j=2}^k 1_{\mathcal S} g_j(a_jn+b_j+h_j) + O\Big(kAX\frac{\log W}{\log H}\Big).\end{aligned} \quad (5.4)$$

Hence the claim follows in the case when $H \le H_0$.

If $H > H_0$, one can cover the summation over the h_j indices by intervals of length H_0 and apply Theorem 1.6 to each subinterval (shifting the b_j by at most AX when doing so), and then sum, noting that the quantity

$$\exp(-M(g_{j_0}; 10AX, Q)/80) + \frac{\log\log H}{\log H} + \frac{1}{\log^{1/3000} X}$$

is essentially unchanged after replacing H with H_0. □

Remark 5.2. By using larger choices of W, one can obtain more refined information on the large values of the correlations $\sum_{1\le n\le X} g_1(a_1n+b_1)\prod_{j=2}^k g_j(a_jn+b_j+h_j)$. For instance, if we take $W = H^\delta$ for some $10 \le H \le H_0$ and $20\frac{\log\log H}{\log H} \le \delta \le \frac{1}{500}$, we see from Proposition 5.1, (5.4), and Markov's inequality that

$$\sum_{1\le n\le X} g_1(a_1n+b_1)\prod_{j=2}^k g_j(a_jn+b_j+h_j) \ll kA^2\delta X$$

for all but at most $O(\frac{H^{k-1}}{\delta H^{\delta/20}})$ tuples $(h_1, \dots, h_{k-1})$ with $1 \le h_j \le H$ for $j = 2, \dots, k$. Thus we can obtain a power saving in the number of exceptional tuples, at the cost of only obtaining a weak bound on the individual correlations $\sum_{1 \le n \le X} g_1(a_1 n + b_1) \prod_{j=2}^k g_j(a_j n + b_j + h_j)$.

It remains to prove Proposition 5.1. We start by proving the following simpler case to which the general case will be reduced.

Proposition 5.3. *Let $X, H, W \ge 10$ be such that*

$$\log^{20} H \le W \le \min\{H^{1/250}, (\log X)^{1/125}\}.$$

Let $g : \mathbb{N} \to \mathbb{C}$ be 1-bounded multiplicative function such that

$$W \le \exp(M(g; X, W)/3).$$

Set

$$\mathcal{S} = \mathcal{S}_{P_1, Q_1, \sqrt{X}, X}$$

where

$$P_1 := W^{200}; \quad Q_1 := H/W^3.$$

Then

$$\sum_{1 \le h \le H} \left| \sum_{1 \le n \le X} 1_{\mathcal{S}} g(n) 1_{\mathcal{S}} \overline{g}(n+h) \right|^2 \ll \frac{HX^2}{W^{1/5}}. \tag{5.5}$$

To deduce Theorem 1.2 we let $\mathcal{S}$ be as in this proposition with $W := H^{\delta/900}$. The argument of Lemma 2.2 actually gives in this case $\#\{1 \le n \le X : n \notin \mathcal{S}\} \le 2\frac{\log P_1}{\log Q_1} X$, and thus the numbers n with $n \notin \mathcal{S}$ or $n+h \notin \mathcal{S}$ contribute to the left hand side of (1.4) at most $9\delta/10$. Hence, recalling (1.11), the claim follows from the previous proposition and Markov's inequality.

Proof of Proposition 5.3. The claim follows once we have shown

$$\sum_{|h| \le 2H} (2H - |h|)^2 \cdot \left| \sum_n 1_{\mathcal{S}} g(n) 1_{\mathcal{S}} \overline{g}(n+h) \right|^2 \ll \frac{1}{W^{1/5}} H^3 X^2.$$

Applying Lemma 1.4, it will suffice to show that

$$\int_{\mathbb{T}} \left(\int_{\mathbb{R}} \left| \sum_{x \le n \le x+2H} 1_{\mathcal{S}} g(n) e(\alpha n) \right|^2 dx \right)^2 d\alpha \ll \frac{1}{W^{1/5}} H^3 X^2.$$

From the Parseval identity we have

$$\begin{aligned} \int_{\mathbb{T}} \int_{\mathbb{R}} \left| \sum_{x \le n \le x+2H} 1_{\mathcal{S}} g(n) e(\alpha n) \right|^2 dx \, d\alpha &= \int_{\mathbb{R}} \sum_{x \le n \le x+2H} |1_{\mathcal{S}} g(n)|^2 \, dx \\ &\ll HX \end{aligned}$$

so it suffices to show that

$$\sup_\alpha \int_{\mathbb{R}} \left| \sum_{x \le n \le x+2H} 1_{\mathcal{S}} g(n) e(\alpha n) \right|^2 dx \ll \frac{1}{W^{1/5}} H^2 X.$$

Using the trivial bound

$$\Big|\sum_{x\leq n\leq x+2H} 1_{\mathcal{S}} g(n) e(\alpha n)\Big| \ll H$$

we thus reduce to showing

$$\sup_\alpha \int_{\mathbb{R}} \Big|\sum_{x\leq n\leq x+2H} 1_{\mathcal{S}} g(n) e(\alpha n)\Big|\ dx \ll \frac{HX}{W^{1/5}}. \tag{5.6}$$

But this follows from Theorem 2.3. □

Proof of Proposition 5.1. We first remove the special treatment afforded to the g_1 factor in (5.1). Note that we may assume that

$$W^{1/20} \geq kA^2 \tag{5.7}$$

and thus

$$H \geq W^{500} \geq (kA^2)^{10000}$$

since the claim is trivial otherwise.

Set $H' := \sqrt{H}$. For any $1 \leq h_1 \leq H'/A$, we may shift n by h_1 and conclude that

$$\begin{aligned}&\sum_{1\leq n\leq X} 1_{\mathcal{S}} g_1(a_1 n + b_1) \prod_{j=2}^k 1_{\mathcal{S}} g_j(a_j n + b_j + h_j)\\ &= \sum_{1\leq n\leq X} 1_{\mathcal{S}} g_1(a_1 n + b_1 + a_1 h_1) \prod_{j=2}^k 1_{\mathcal{S}} g_j(a_j n + b_j + h_j + a_j h_1) + O(H')\end{aligned}$$

and thus we may write the left-hand side of (5.1) as

$$\sum_{1\leq h_2,\dots,h_k\leq H} \Big|\sum_{1\leq n\leq X} 1_{\mathcal{S}} g_1(a_1 n + b_1 + a_1 h_1) \prod_{j=2}^k 1_{\mathcal{S}} g_j(a_j n + b_j + h_j + a_j h_1)\Big| + O(H^{k-1} H').$$

If one shifts each of the h_j for $j = 2, \dots, k$ in turn by $a_j h_1 = O(H')$, we may rewrite this as

$$\sum_{1\leq h_2,\dots,h_k\leq H} \Big|\sum_{1\leq n\leq X} 1_{\mathcal{S}} g_1(a_1 n + b_1 + a_1 h_1) \prod_{j=2}^k 1_{\mathcal{S}} g_j(a_j n + b_j + h_j)\Big| + O(H^{k-1} H') + O(kH^{k-2} H' X).$$

Averaging in h_1, and replacing h_1 by $a_1 h_1$ (crudely dropping the constraint that $a_1 h_1$ is divisible by a_1), we may thus bound the left-hand side of (5.1) by

$$\begin{aligned}\ll \frac{A}{H'} \sum_{1\leq h_1\leq H'} \sum_{1\leq h_2,\dots,h_k\leq H} \Big|\sum_{1\leq n\leq X} 1_{\mathcal{S}} g_1(a_1 n + b_1 + h_1) \prod_{j=2}^k 1_{\mathcal{S}} g_j(a_j n + b_j + h_j)\Big| +\\ + H^{k-1} H' + kH^{k-2} H' X.\end{aligned}$$

The g_1 term may now be combined with the product over the remaining g_j terms to form $\prod_{j=1}^k 1_{\mathcal{S}} g_j(a_j n + b_j + h_j)$. The error term $H^{k-1} H' + kH^{k-2} H' X$ is certainly of size $O(\frac{kA^2}{W^{1/20}} H^{k-1} X)$, so it suffices to show that

$$\sum_{1\leq h_1\leq H'} \sum_{1\leq h_2,\dots,h_k\leq H} \Big|\sum_{1\leq n\leq X} \prod_{j=1}^k 1_{\mathcal{S}} g_j(a_j n + b_j + h_j)\Big| \ll \frac{A}{W^{1/20}} H^{k-1} H' X.$$

By covering the ranges $1 \leq h_j \leq H$ by intervals of length H' and averaging, it suffices (after relaxing the conditions $b_j \leq 3AX$ to $b_j \leq 4AX$) to prove that

$$\sum_{1 \leq h_1, h_2, \dots, h_k \leq H'} \left| \sum_{1 \leq n \leq X} \prod_{j=1}^{k} 1_{\mathcal{S}} g_j(a_j n + b_j + h_j) \right| \ll \frac{A}{W^{1/20}} (H')^k X.$$

By the Cauchy-Schwarz inequality, it suffices to show that

$$\sum_{1 \leq h_1, h_2, \dots, h_k \leq H'} \left| \sum_{1 \leq n \leq X} \prod_{j=1}^{k} 1_{\mathcal{S}} g_j(a_j n + b_j + h_j) \right|^2 \ll \frac{A^2}{W^{1/10}} (H')^k X^2.$$

The left-hand side may be rewritten as

$$\sum_{n, n' \leq X} \prod_{j=1}^{k} \sum_{1 \leq h_j \leq H'} 1_{\mathcal{S}} g_j(a_j n + b_j + h_j) 1_{\mathcal{S}} \overline{g_j}(a_j n' + b_j + h_j).$$

Recall we have a specific index j_0 in Theorem 1.6. For $j \neq j_0$, we crudely bound

$$\left| \sum_{1 \leq h_j \leq H'} 1_{\mathcal{S}} g_j(a_j n + b_j + h_j) 1_{\mathcal{S}} \overline{g_j}(a_j n' + b_j + h_j) \right| \leq H'$$

so it suffices to show that

$$\sum_{n, n' \leq X} \left| \sum_{1 \leq h_{j_0} \leq H'} 1_{\mathcal{S}} g_{j_0}(a_{j_0} n + b_{j_0} + h_{j_0}) 1_{\mathcal{S}} \overline{g_{j_0}}(a_{j_0} n' + b_{j_0} + h_{j_0}) \right| \ll \frac{A^2}{W^{1/10}} H' X^2.$$

To abbreviate notation we now write $h = h_{j_0}$, $g = g_{j_0}$, $a = a_{j_0}$, $b = b_{j_0}$. By the Cauchy-Schwarz inequality, it suffices to show that

$$\sum_{n, n' \leq X} \left| \sum_{1 \leq h \leq H'} 1_{\mathcal{S}} g(an + b + h) 1_{\mathcal{S}} \overline{g}(an' + b + h) \right|^2 \ll \frac{A^4}{W^{1/5}} (H')^2 X^2.$$

Replacing n, n' by $an + b$, $an' + b$ respectively, it suffices to show that

$$\sum_{n, n'} \left| \sum_{1 \leq h \leq H'} 1_{\mathcal{S}} g(n + h) 1_{\mathcal{S}} \overline{g}(n' + h) \right|^2 \ll \frac{A^4}{W^{1/5}} (H')^2 X^2$$

where we have extended $1_{\mathcal{S}} g$ by zero to the negative integers. The left-hand side can be rewritten as

$$\sum_{|h| < H'} (\lfloor H' \rfloor - |h|) \left| \sum_{n} 1_{\mathcal{S}} g(n) 1_{\mathcal{S}} \overline{g}(n + h) \right|^2,$$

and the claim follows from Proposition 5.3. □

Appendix A. Mean values of complex multiplicative functions in short intervals

In this section we prove a complex variant of results in [14] in case f is not p^{it} pretentious. In particular we show that, for almost all short intervals, the mean value of a 1-bounded nonpretentious multiplicative function is zero:

Theorem A.1. *Let f be a 1-bounded multiplicative function and let $M(f;X)$ be as in* (1.5). *Then, for $X \geq h \geq 10$,*

$$\frac{1}{X}\int_X^{2X}\left|\frac{1}{h}\sum_{x\leq n\leq x+h} f(n)\right|^2 dx \ll \exp(-M(f;X))M(f;X) + \frac{(\log\log h)^2}{(\log h)^2} + \frac{1}{(\log X)^{1/50}}.$$

Actually as in [14] and earlier in this paper, one gets better quantiative results if one first restricts to a subset of n with a typical factorization. Let us first define such subset $\mathcal{S}$ in this setting.

Let $\eta \in (0,1/6)$, and let X_0 be a quantity with $\sqrt{X} \leq X_0 \leq X$. (The results in [14] used the choice $X_0 = X$, but for technical reasons we will need a more flexible choice of this parameter.) Consider a sequence of increasing intervals $[P_j,Q_j], j \geq 1$ such that

- $Q_1 \leq \exp(\sqrt{\log X_0})$.
- The intervals are not too far from each other, precisely
$$\frac{\log\log Q_j}{\log P_{j-1} - 1} \leq \frac{\eta}{4j^2} \tag{A.1}$$
for all $j \geq 2$.
- The intervals are not too close to each other, precisely
$$\frac{\eta}{j^2}\log P_j \geq 8\log Q_{j-1} + 16\log j \tag{A.2}$$
for all $j \geq 2$.

For example, given $0 < \eta < 1/6$, the sequence of intervals $[P_j,Q_j]$ defined in Definition 2.1 can be verified to obey the above estimates if

$$\exp(\sqrt{\log X_0}) \geq Q_1 \geq P_1 \geq (\log Q_1)^{40/\eta}$$

and P_1 is sufficiently large.

Let $\mathcal{S}$ be the set of integers $X \leq n \leq 2X$ having at least one prime factor in each of the intervals $[P_j,Q_j]$ for $j \leq J$, where J is chosen to be the largest index j such that $Q_j \leq \exp((\log X_0)^{1/2})$. We will establish the following variant of [14, Theorem 3].

Theorem A.2. *Let f be a 1-bounded multiplicative function. Let $\mathcal{S}$ be as above with $\eta \in (0,1/6)$. If $[P_1,Q_1] \subset [1,h]$, then for all $X > X(\eta)$ large enough and $h \geq 3$,*

$$\frac{1}{X}\int_X^{2X}\left|\frac{1}{h}\sum_{\substack{x\leq n\leq x+h\\ n\in\mathcal{S}}} f(n)\right|^2 dx \ll \exp(-M(f;X))M(f;X) + \frac{(\log h)^{1/3}}{P_1^{1/6-\eta}} + \frac{1}{(\log X)^{1/50}}.$$

The proof of Theorem A.2 proceeds as the proof of [14, Theorem 3]. The first step is a Parseval bound

$$\frac{1}{X}\int_X^{2X}\left|\frac{1}{h}\sum_{\substack{x\leq n\leq x+h\\ n\in\mathcal{S}}} f(n)\right|^2 dx \ll \int_1^{1+iX/h_1}|F(s)|^2\,|ds| + \max_{T\geq X/h_1}\frac{X/h_1}{T}\int_{1+iT}^{1+i2T}|F(s)|^2\,|ds|.$$

This follows exactly in the same way as [14, Lemma 14] but there is no need to split the integral into two parts, and one can just work as for $V(x)$ there. Theorem A.2 now follows immediately from the following variant of [14, Proposition 1].

Proposition A.3. *Let f be a 1-bounded multiplicative function. Let $\mathcal{S}$ be as above, and let*

$$F(s) = \sum_{\substack{X \le n \le 2X \\ n \in \mathcal{S}}} \frac{f(n)}{n^s}.$$

Then, for any T,

$$\int_0^T |F(1+it)|^2 \, dt \ll \left(\frac{T}{X/Q_1} + 1 \right) \left(\frac{(\log Q_1)^{1/3}}{P_1^{1/6-\eta}} + \exp(-M(f;X))M(f;X) + \frac{1}{(\log X)^{1/50}} \right).$$

Proof. Since the mean value theorem gives the bound $O(T/X + 1)$, we can assume $T \le X/2$ and $M(f;X) \ge 1$.

Let now t_1 be the value of t which attains the minimum in

$$M(f;X) = \inf_{|t| \le X} \mathbb{D}(g, n \mapsto n^{it}; X)^2.$$

We split the integration into three ranges:

$$\mathcal{T}_0 = \{0 \le t \le T \colon |t - t_1| \le \exp(M(f;X))/M(f;X)\}$$
$$\mathcal{T}_1 = \{0 \le t \le T \colon \exp(M(f;X))/M(f;X) \le |t - t_1| \le (\log X)^{1/16}\}$$
$$\mathcal{T}_2 = \{0 \le t \le T \colon |t - t_1| \ge (\log X)^{1/16}\}.$$

Notice that by the definition of t_1, the triangle inequality and arguing as in (1.11), for any $|t| \le X$ with $|t - t_1| \ge 1$, and any $\varepsilon > 0$,

$$2\mathbb{D}(f, p^{it}; X) \ge \mathbb{D}(f, p^{it}; X) + \mathbb{D}(f, p^{it_1}; X) \ge \mathbb{D}(1, p^{i(t-t_1)}) \ge \left(\frac{1}{\sqrt{3}} - \varepsilon \right) \sqrt{\log \log X} + O(1),$$

so that by Halasz's theorem, for every $|t| \le T$,

$$F(1+it) \ll (\log X)^{-1/16} + \frac{1}{1 + |t - t_1|}.$$

In the region $|t - t_1| \ge (\log X)^{1/16}$, the above implies the following in exactly the same way as [14, Lemma 3].

Lemma A.4. *Let $X \ge Q \ge P \ge 2$. Let t_1 be as above and*

$$G(s) = \sum_{X \le n \le 2X} \frac{f(n)}{n^s} \cdot \frac{1}{\#\{p \in [P,Q] \colon p \mid n\} + 1}.$$

Then, for any $t \in \mathcal{T}_2$,

$$|G(1+it)| \ll \frac{\log Q}{(\log X)^{1/16} \log P} + \log X \cdot \exp\left(-\frac{\log X}{3 \log Q} \log \frac{\log X}{\log Q} \right).$$

This was the only part in the proof [14, Proposition 1] that needed f to be real-valued and thus we get

$$\int_{\mathcal{T}_2} |F(1+it)|^2 \, dt \ll \left(\frac{T}{X/Q_1} + 1 \right) \left(\frac{(\log Q_1)^{1/3}}{P_1^{1/6-\eta}} + \frac{1}{(\log X)^{1/50}} \right).$$

Using the estimate $F(1+it) \ll \frac{1}{|t-t_1|}$ for $t \in \mathcal{T}_1$ and the estimate $F(1+it) \ll \exp(-M(f;X))M(f;X)$ coming from Halasz's theorem for $t \in \mathcal{T}_0$, we obtain

$$\int_{\mathcal{T}_0 \cup \mathcal{T}_1} |F(1+it)|^2 \, dt \ll \exp(-M(f;X))M(f;X),$$

and the claim follows. $\square$

Proof of Theorem A.1. Let $\eta = 1/12$, $P_1 = (\log h)^{480}, Q_1 = h$, let P_j and Q_j for $j \geq 2$ be as in Definition 2.1, and let $\mathcal{S}$ be as above. Then

$$\frac{1}{X}\int_X^{2X} \left| \frac{1}{h} \sum_{x \leq n \leq x+h} f(n) \right|^2 dx \leq \frac{1}{X}\int_X^{2X} \left| \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \in \mathcal{S}}} f(n) \right|^2 dx + \frac{1}{X}\int_X^{2X} \left| \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \notin \mathcal{S}}} 1 \right|^2 dx.$$

The contribution from the first integral is acceptable by Theorem A.2. We rewrite the second integrand as

$$\begin{aligned} \left| \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \notin \mathcal{S}}} 1 \right| &= \left| 1 + O(1/h) - \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \in \mathcal{S}}} 1 \right| \\ &\leq \left| \frac{1}{X} \sum_{\substack{X \leq n \leq 2X \\ n \in \mathcal{S}}} 1 - \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \in \mathcal{S}}} 1 \right| + \left| \frac{1}{X} \sum_{\substack{X \leq n \leq 2X \\ n \notin \mathcal{S}}} 1 \right| + O(1/h), \end{aligned}$$

and the claim follows from [14, Theorem 3 with $f = 1$] and Lemma 2.2. $\square$

Appendix B. Counterexample to the uncorrected Elliott conjecture

In this appendix we present a counterexample to Conjecture 1.5. More precisely:

Theorem B.1 (Counterexample)**.** *There exists a 1-bounded multiplicative function $g : \mathbb{N} \to \mathbb{C}$ such that*

$$\sum_p \frac{1 - \mathrm{Re}(g(p)\overline{\chi(p)}p^{-it})}{p} = \infty \tag{B.1}$$

for all Dirichlet characters χ and $t \in \mathbb{R}$ (i.e., one has $M(g;\infty,\infty) = \infty$), but such that

$$\left| \sum_{n \leq t_m} g(n)\overline{g(n+1)} \right| \gg t_m \tag{B.2}$$

for all sufficiently large m, and some sequence t_m going to infinity.

Proof. For each prime p, we choose $g(p)$ from the unit circle $S^1 := \{z : |z| = 1\}$ by the following iterative procedure involving a sequence $t_1 < t_2 < t_3 < \dots$.

(1) Initialize $t_1 := 100$ and $m := 1$, and set $g(p) := 1$ for all $p \leq t_1$.
(2) Now suppose recursively that $g(p)$ has been chosen for all $p \leq t_m$. As the quantities $\log p$ are linearly independent over the integers, the sequence $t \mapsto (t \log p \bmod 1)_{p \leq t_m}$ is equidistributed in the torus $\prod_{p \leq t_m} \mathbb{T}$; equivalently, the

sequence $t \mapsto (p^{it})_{p \leq t_m}$ is equidistributed in the torus $\prod_{p \leq t_m} S^1$. Thus one can find a quantity $s_{m+1} > \exp(t_m)$ such that

$$p^{is_{m+1}} = g(p)\left(1 + O\left(\frac{1}{t_m^2}\right)\right) \tag{B.3}$$

for all $p \leq t_m$.

(3) Set $t_{m+1} := s_{m+1}^2$, and then set

$$g(p) := p^{is_{m+1}} \tag{B.4}$$

for all $t_m < p \leq t_{m+1}$. Now increment m to $m+1$ and return to step 2.

Clearly the t_m go to infinity, so $g(p)$ is defined for all primes p. We then define

$$g(n) := \mu(n)^2 \prod_{p|n} g(p), \tag{B.5}$$

which is clearly a 1-bounded multiplicative function.

Suppose that $n \leq t_{m+1}$ is squarefree. Then n is the product of distinct primes less than or equal to t_{m+1}, including at most t_m primes less than or equal to t_m. From (B.5) we then have

$$\begin{aligned} g(n) &= n^{is_{m+1}}\left(1 + O\left(\frac{1}{t_m^2}\right)\right)^{O(t_m)} \\ &= n^{is_{m+1}} + O\left(\frac{1}{t_m}\right). \end{aligned}$$

If n is not squarefree, then $g(n)$ of course vanishes. We thus have, for $t_{m+1}^{3/4} \leq n \leq t_{m+1} - 1$,

$$\begin{aligned} g(n)\overline{g(n+1)} &= \mu^2(n)\mu^2(n+1)\left(\frac{n+1}{n}\right)^{is_{m+1}} + O\left(\frac{1}{t_m}\right) \\ &= \mu^2(n)\mu^2(n+1) + O\left(\frac{s_{m+1}}{t_{m+1}^{3/4}}\right) + O\left(\frac{1}{t_m}\right) \\ &= \mu^2(n)\mu^2(n+1) + O\left(\frac{1}{t_m}\right), \end{aligned}$$

and the claim (B.2) then easily follows since the sequence $\mu^2(n)\mu^2(n+1)$ has positive mean value.

Now we prove (B.1). From (B.4), we have

$$\begin{aligned} \sum_p \frac{1 - \mathrm{Re}(g(p)\overline{\chi(p)}p^{-it})}{p} &\geq \sum_{t_m < p \leq t_{m+1}} \frac{1 - \mathrm{Re}(\overline{\chi(p)}p^{i(s_{m+1}-t)})}{p} \\ &\geq \sum_{\exp((\log t_{m+1})^{5/6}) < p \leq t_{m+1}} \frac{1 - \mathrm{Re}(\overline{\chi(p)}p^{i(s_{m+1}-t)})}{p} \end{aligned}$$

since $\exp((\log t_{m+1})^{5/6}) \geq \exp((2t_m)^{5/6}) \geq t_m$. Hence we see as in (1.11) that the right-hand side goes to infinity as $m \to \infty$ for any fixed χ, t, and the claim follows. $\square$

It is easy to see that the function g constructed in the above counterexample violates (1.8), and so is not a counterexample to the corrected form of Conjecture 1.5. It is also not difficult to modify the above counterexample so that the function g is completely multiplicative instead of multiplicative, using the fact that most numbers up to t_{m+1} have fewer than t_m prime factors less than t_m (counting multiplicity); we leave the details to the interested reader.

Appendix C. An argument of Granville and Soundararajan

In this appendix we show the equivalence of the hypotheses (1.6) and (1.8) for Elliott's conjecture in the case that the multiplicative function g_{j_0} is real. The key lemma is the following estimate, essentially due to Granville and Soundararajan.

Lemma C.1. *Let $f\colon \mathbb{N} \to [-1,1]$ be a multiplicative function, let $x \geq 100$, and let χ be a fixed Dirichlet character. For $1 \leq |\alpha| \leq x$, one has*

$$\mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x) \geq \frac{1}{4}\sqrt{\log\log x} + O_\chi(1). \tag{C.1}$$

When χ^2 is non-principal, this holds for all $|\alpha| \leq x$.

If χ^2 is principal (i.e., χ is a quadratic character), then, for $|\alpha| \leq 1$, one has

$$\mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x) \geq \frac{1}{3}\mathbb{D}(f, \chi; x) + O(1). \tag{C.2}$$

Proof. To establish (C.1), we notice that, by conjugation symmetry and the triangle inequality,

$$\begin{aligned}
\mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x) &= \frac{1}{2}(\mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x) + \mathbb{D}(f, n \mapsto \overline{\chi(n)}n^{-i\alpha}; x))\\
&\geq \frac{1}{2}\mathbb{D}(n \mapsto \overline{\chi(n)}n^{-i\alpha}, n \mapsto \chi(n)n^{i\alpha}; x)\\
&= \frac{1}{2}\left(\sum_{p \leq x} \frac{1 - \operatorname{Re}\chi^2(p)p^{2i\alpha}}{p}\right)^{1/2}
\end{aligned}$$

which implies the claim for $|\alpha| \geq 1$ or for non-principal χ^2 by the zero-free (and pole-free) region for Dirichlet L-functions (see (1.11) for a related argument).

To establish (C.2), notice first that since χ^2 is principal, χ is real-valued which implies together with the triangle inequality

$$\mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x) = \mathbb{D}(f\chi, n \mapsto n^{i\alpha}; x) \geq \mathbb{D}(1, f\chi; x) - \mathbb{D}(1, n \mapsto n^{i\alpha}; x).$$

Now $\mathbb{D}(1, n \mapsto n^{i\alpha}; x) = \mathbb{D}(1, n \mapsto n^{2i\alpha}; x) + O(1)$ for $|\alpha| \leq 1$, since $\mathbb{D}(1, n \mapsto n^{i\alpha}; x)^2 = \log(1+|\alpha|\log x)+O(1)$ from the prime number theorem, so that the claim follows unless $\mathbb{D}(1, n \mapsto n^{2i\alpha}; x) \geq \frac{2}{3}\mathbb{D}(1, f\chi; x)$. But in the latter case, the triangle inequality gives

$$\begin{aligned}
\frac{2}{3}\mathbb{D}(f, \chi; x) &= \frac{2}{3}\mathbb{D}(1, f\chi; x)\\
&\leq \mathbb{D}(1, n \mapsto n^{2i\alpha}; x)\\
&= \mathbb{D}(n \mapsto n^{-i\alpha}, n \mapsto n^{i\alpha}; x)\\
&\leq \mathbb{D}(f\chi, n \mapsto n^{-i\alpha}; x) + \mathbb{D}(f\chi; n \mapsto n^{i\alpha}; x)\\
&= 2\mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x),
\end{aligned}$$

and the claim (C.2) follows. □

From this lemma, we see that when g_{j_0} is a real 1-bounded multiplicative function, then for given Q, the condition (1.8) is equivalent to

$$\mathbb{D}(g_{j_0}, \chi; X) \to \infty$$

when $X \to \infty$ for all quadratic characters χ of modulus at most Q. But this follows from (1.6). The converse implication is trivial.

References

[1] Antal Balog. The prime k-tuplets conjecture on average. In *Analytic number theory (Allerton Park, IL, 1989)*, volume 85 of *Progr. Math.*, pages 47–75. Birkhäuser Boston, Boston, MA, 1990.

[2] J. Bourgain, P. Sarnak, and T. Ziegler. Disjointness of Moebius from horocycle flows. In *From Fourier analysis and number theory to radon transforms and geometry*, volume 28 of *Dev. Math.*, pages 67–83. Springer, New York, 2013.

[3] S. Chowla. *The Riemann hypothesis and Hilbert's tenth problem.* Mathematics and Its Applications, Vol. 4. Gordon and Breach Science Publishers, New York-London-Paris, 1965.

[4] Hédi Daboussi and Hubert Delange. On multiplicative arithmetical functions whose modulus does not exceed one. *J. London Math. Soc. (2)*, 26(2):245–264, 1982.

[5] Harold Davenport. On some infinite series involving arithmetical functions. ii. *Quart. J. Math. Oxf.*, 8:313–320, 1937.

[6] P. D. T. A. Elliott. On the correlation of multiplicative functions. *Notas Soc. Mat. Chile*, 11(1):1–11, 1992.

[7] J. Friedlander and H. Iwaniec. *Opera de cribro*, volume 57 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2010.

[8] Andrew Granville and K. Soundararajan. Large character sums: pretentious characters and the Pólya-Vinogradov theorem. *J. Amer. Math. Soc.*, 20(2):357–384, 2007.

[9] Benjamin Green and Terence Tao. Linear equations in primes. *Ann. of Math. (2)*, 171(3):1753–1850, 2010.

[10] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, Rhode Island, 2004.

[11] I. Kátai. A remark on a theorem of H. Daboussi. *Acta Math. Hungar.*, 47(1-2):223–225, 1986.

[12] K. Kawada. A Montgomery-Hooley type theorem for prime k-tuplets. *Acta Math. Hungar.*, 66(3):177–200, 1995.

[13] Koichi Kawada. The prime k-tuplets in arithmetic progressions. *Tsukuba J. Math.*, 17(1):43–57, 1993.

[14] K. Matomäki and M. Radziwiłł. Multiplicative functions in short intervals. *Preprint*, available at `http://arxiv.org` as arXiv:1501.04585 [math.NT].

[15] Hiroshi Mikawa. On prime twins in arithmetic progressions. *Tsukuba J. Math.*, 16(2):377–387, 1992.

[16] H. L. Montgomery and R. C. Vaughan. Exponential sums with multiplicative coefficients. *Invent. Math.*, 43(1):69–82, 1977.

[17] Hugh L. Montgomery. *Ten lectures on the interface between analytic number theory and harmonic analysis*, volume 84 of *CBMS Regional Conference Series in Mathematics.* Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994.

Department of Mathematics and Statistics, University of Turku, 20014 Turku, Finland

E-mail address: ksmato@utu.fi

Department of Mathematics, Rutgers University, Hill Center for the Mathematical Sciences, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019

E-mail address: maksym.radziwill@gmail.com

Department of Mathematics, UCLA, 405 Hilgard Ave, Los Angeles CA 90095, USA

E-mail address: tao@math.ucla.edu