Homework VII (Honours) Analysis III

Tuesday 22$^{\text{nd}}$ November, 2016

1 Problems

1.1 Problem I

Proposition not true as Maksym communicated through e-mail. Though one should remark that it is true if we assume that $\mathcal{M}$ is an algebra. That is, if we assume in addition that $\mathcal{M}$ is closed under finite (not necessarily disjoint) unions. It seems that Stein and Shakarchi forgot to include this condition when they changed terminology between two editions, to remove dependency on the term "algebra" in the question. Which makes sense because the term is introduced later than the theorem where this exercise is relevant (actually necessary, but for some reason they also forgot to mention this). See Theorem 1.2 (p. 267).

1.2 Problem II

Proposition (Caratheodory's criterion): If λ_* denotes the Lebesgue exterior measure then a set E is λ_*-Caratheodory measurable if and only if E is Lebesgue measurable.

Proof: "$\Leftarrow$": Since every Lebesgue measurable sets E can be written as $E = G - Z$, with $G \in G_\delta$ and $\lambda_*(Z) = 0$, and the set of λ_*-Caratheodory measurable sets is a σ-algebra, it suffices to show that G and Z, chosen arbitrary, are λ_*-Caratheodory measurable sets. For Z this claim is immediate from the definition of λ_*-Caratheodory measurability. Now, per assumption, $G = \bigcap_i^\infty O_i$ with O_i open sets. Therefore if we show that arbitrary open sets are λ_*-Caratheodory measurable sets, we are done (again because λ_*-Caratheodory measurable sets form a σ-algebra, hence are closed under countable intersection). Or even simpler, because any open set can be written as a countable union of open intervals (we assume that we are in $\mathbb{R}$ but for $\mathbb{R}^d$ the proof is very similar but then using a countable union of closed rectangles), we are done if we can show λ_*-Caratheodory measurability of arbitrary open intervals $O = \bigcup_i^\infty I_i$. Put different, it is enough to show that for any set A, $\lambda_*(A) \geq \lambda_*(I \cap A) + \lambda_*(I^c \cap A)$, where $I = (a, b)$ (the other inequality follows from countable additivity of Lebesgue exterior measure).

To see why this last inequality holds, take any rectangle covering of A, $\{Q_i\}$. Per definition of the Lebesgue exterior measure $\lambda_*(A) = \inf \sum_i^\infty |Q_i|$, with the infimum taken over all such rectangle coverings. We then split every $Q_i = [c, d]$ that intersects (a, b) into (at most) three parts (assuming without loss of generalization that $c < d$):

- $Q_{i,1} = [c, a]$ and $Q_{i,2} = [a, d]$, if the intersection is only at a.
- $Q_{i,1} = [c, b]$ and $Q_{i,2} = [b, d]$, if the intersection is only at b.
- $Q_{i,1} = [c, a]$, $Q_{i,2} = [a, b]$ and $Q_{i,3} = [b, d]$, if the intersection is at both a and b.

In any case, however, $|Q_i| = |Q_{i,1}| + |Q_{i,2}| + |Q_{i,3}|$ (taking $Q_{i,3} = \emptyset$ when appropriate). Therefore $\sum_{i,j} |Q_{i,j}| = \sum_i |Q_i|$. But now notice that $\{Q_{i,j}\}$ can partitioned into rectangle covers $\{X_i\}$ and $\{Y_i\}$ for $I \cap A$ and $I^c \cap A$ respectively. So that $\sum_i |Q_i| = \sum_{i,j} |Q_{i,j}| = \sum_i |X_i| + \sum_i |Y_i| \geq \lambda_*(I \cap A) + \lambda_*(I^c \cap A)$, and therefore since the covering $\{Q_i\}$ was chosen arbitrary $\lambda_*(A) \geq \lambda_*(I \cap A) + \lambda_*(I^c \cap A)$. And we are done in this direction.

"$\Rightarrow$": Let E be an arbitrary λ_*-Caratheodory measurable set. Pick $\{O_n\}$ open sets such that $O_n \supset E$ and $\lambda_*(O_n) \leq \lambda_*(E) + 1/n$, and let the G_δ set $G = \bigcap_n^\infty O_n$. Now because still $G \supset E$ we have $\lambda_*(E) = \lambda_*(G)$. Also, since E is assumed λ_*-Caratheodory measurable, $\lambda_*(G) = \lambda_*(E \cap G) + \lambda_*(E^c \cap G) = \lambda_*(E) + \lambda_*(G - E) \Rightarrow \lambda_*(G - E) = 0$. But G is Lebesgue measurable, hence there is an open set $O \supset G$ such that for every $\epsilon > 0$ we have $\lambda_*(O - G) < \epsilon$. Therefore for every $\epsilon > 0$ we have also $\lambda_*(O - E) = \lambda_*(O - G \cup G - E) \leq \lambda_*(O - G) + \lambda_*(G - E) = \lambda(O - G) < \epsilon \Rightarrow \lambda_*(O - E) < \epsilon$. But this is precisely the criterion for Lebesgue measurability of E, hence we are done in this direction too and the proposition is established.

$\square$

1.3 Problem III

(a) If A is disjoint from C and B is disjoint from C, then also $A \cup B$ is disjoint from C. Take A to be the support of ν_1, B the support of ν_2 and C the support of μ, and we are done, as $A \cup B$ is the support of $\nu_1 + \nu_2$.

(b) Take $E \in \mathcal{M}$, the σ-algebra associated with μ. Then per assumption if $\mu(E) = 0$ we have $\nu_1(E) = 0$ and $\nu_2(E) = 0$. But also $(\nu_1 + \nu_2)(E) = \nu_1(E) + \nu_2(E) = 0$, thus $\nu_1 + \nu_2 \ll \mu$.

(c) This is proposition 4.1, p. 286. But if one insists on the part of the proof that is not there explicitly, take any $E \in \mathcal{M}$ and any $\mathcal{M}$-partition of that E, $E = \bigcup_i^\infty E_i$, then $\nu(E) = \nu(\bigcup_i^\infty E_i) = \sum_i^\infty \nu(E_i)$. But $|\nu|(E) = \sup \sum_j^\infty |\nu(E_j)| \geq \sup \sum_j^\infty \nu(E_j) \geq \sum_i^\infty \nu(E_i) = \nu(E)$, because the $\{E_i\}$ $\mathcal{M}$-partition is one of those over which the supremum is taken. [Actually now that I think of it, it is even simpler because E is a $\mathcal{M}$-partition of itself, and certainly $|\nu(E)| \geq \nu(E)$...]

(d) Take any $E \in \mathcal{M}$ and let A and B denote the support of μ and ν respectively. Since $\nu \perp \mu$, $A \cap B = \emptyset$. Now $\nu(E \cap A^c) = 0$, because $\mu(E \cap A^c) = \mu(E \cap A^c \cap A) = \mu(\emptyset) = 0$ and $\nu \ll \mu$, but also $\nu(E \cap A) = \nu(E \cap A \cap B) = \nu(\emptyset) = 0$, and therefore $\nu(E) = \nu(E \cap A^c) + \nu(E \cap A) = 0$. Because E was taken arbitrary, we conclude $\nu = 0$. (Notice that this is indeed a valid measure, though a bit silly one)

1.4 Problem IV

We shall assume also that F is normalized, because otherwise μ_{F_J} is ill-defined. Let μ denote the Lebesgue measure.

(a) It suffices to show that $\mu_A(E) = \int_E F'(x)dx$. Because of the basic observation that integrals over Lebesgue measure zero sets are zero, it would then immediately follow that $\mu_A \ll \mu$. Now because μ_A is unique, since F_A is also an increasing and right-continuous function, if we show that 1) $F'(x)$ is well-defined almost everywhere and measurable, and 2) $\int_a^b F'(x)dx = F_A(b) - F_A(a)$, we would be done [it is clearly a measure if 1) holds]. 1) is by Theorem 3.4 (p. 121) and Corollary 3.7 (p. 125). Now for 2) notice that in the Lebesgue decomposition of F, $F'_C = 0$ almost everywhere (per assumption on F_C) and $F'_J = 0$ almost everywhere (by Theorem 3.14, p. 133), so that $F' = F'_A$ almost everywhere, and hence $\int_a^b F'(x)dx = \int_a^b (F'_A(x) + F'_C(x) + F'_J(x))\,dx = \int_a^b F'_A(x)dx$. But then we can use the Lebesgue variant of the Fundamental Theorem of Calculus, Theorem 3.11 (p. 130), as F_A is absolutely continuous per assumption. That is, since $\int_a^b F'(x)dx = \int_a^b F'_A(x)dx$, and by the Fundamental Theorem of Calculus $\int_a^b F'_A(x)dx = F_A(b) - F_A(a)$, $\int_a^b F'(x)dx = F_A(b) - F_A(a)$, and we have shown 2) and thus are done.

(b) By exercise 3(a) it suffices to show that $\mu_C \perp \mu$ and $\mu_J \perp \mu$. The latter is easily shown by the definition of F_J and $\mu_J((a,b]) = F_J(b) - F_J(a)$, since F_J is constant almost everywhere. That is, the support of μ_J is precisely the collection of those points at which the function jumps. To see this, suppose there is a jump at point x_n of size α_n. Then if we choose $k \geq K$ for some large enough K, we can define the family of sets $\{E_k\}_{k \geq K}$ like $E_k = (x_n - 1/k, x_n]$, all of which have the property that $\mu_J(E_k) = \alpha_n < \infty$. Also $E = \bigcap_k E_k = \{x_n\}$, and therefore by Corollary 3.3(ii) (p. 20) (for general measures) $\mu_J(E) = \mu_J(\{x_n\}) = \lim_{k\to\infty} \mu_J(E_k) = \alpha_n$. From this it then follows that $\mu_J((a,b])$ equals the sum of the α_n jump sizes associated with all jumps $\{x_k\} \subset \{x_n\}$ where $x_k \in (a,b]$, granted a does not coincide with a jump. And more generally that $\mu_J(E)$ equals the sum of the α_n jump sizes associated with all jumps $\{x_k\} \subset \{x_n\}$ where $x_k \in E$.[1] Clearly then it follows that the support of μ_J are just the jump points $\{x_n\}$. Finally, we note that a monotone function, which F_J is, can only have countably many jumps (so that out indexing of $\{x_n\}$ was indeed justified). However, a countable set has zero Lebesgue measure, and is thus irrelevant for the support of μ. Hence indeed $\mu_J \perp \mu$.

Now we show $\mu_C \perp \mu$ analogously. The only difference is that we isolate the measure zero set of points "at" which $F'_C \neq 0$, call it Z. More precisely, let the collection of sets E_n be defined as the rectangle coverings of Z such that $\lambda(E_n) \leq 1/n$ (this we can do because $\lambda(Z) = 0$). We now show that

[1] To see this just consider covering E with some collection of conveniently chosen intervals $(a_i, b_i]$ (that is $a_i \notin \{x_n\}$) and use monotonicity and countable additivity.

every E_n is in fact a support for μ_C, so that $E = \bigcap_n E_n$ is a support for μ_C, where $\lambda(E) = 0$, like we want. But this is easy. Since $E_n \in F_\sigma$, $E_n^c \in G_\delta$, which is Lebesgue measurable, so that there exists for every $\epsilon > 0$ an open set $O_\epsilon \supset E_n^c$ such that $\lambda(O_\epsilon - E_n^c) < \epsilon \Rightarrow \lambda(O_\epsilon \cap E_n) < \epsilon$. Therefore