Homework VI (Honours) Analysis III

Tuesday 1st November, 2016

1 Problems

1.1 Problem I

We must show that Dini function $D^+(F)(x) = \lim_{\delta \to 0} \sup_{0<h<\delta} \frac{F(x+h)-F(x)}{h} = \lim_{n\to\infty} \sup_{0<h<\frac{1}{n}} \frac{F(x+h)-F(x)}{h}$ is a measurable. In particular then, we are done if we show that $\forall \delta > 0$, $\sup_{0<h<\delta} \frac{F(x+h)-F(x)}{h}$ is measurable. Suppose therefore for arbitrary $\delta > 0$ and α that $x_0 \in F_{\delta,\alpha} = \{x : \sup_{0<h<\delta} \frac{F(x+h)-F(x)}{h} > \alpha\}$, we show that there exists some ball containing x_0 that is entirely contained in $F_{\delta,\alpha}$, then $F_{\delta,\alpha}$ is open and hence measurable, and we are done.

For x_0 we have some $h_0 < \delta$ such that $\frac{F(x_0+h_0)-F(x_0)}{h_0} > \alpha$. Now consider the open interval $O = \{x : |x| < \delta_O\}$, by continuity taking δ_O small enough we have for any $\epsilon > 0$, however small, $\forall o \in O$ simultaneously that $F(x_0 + o + h_0) - F(x_0 + h_0) > -\epsilon/2$ and $F(x_0 + o) - F(x_0) < \epsilon/2$, and therefore

$$\begin{aligned}
&\frac{F(x_0 + o + h_0) - F(x_0 + o)}{h_0} \\
=&\frac{(F(x_0 + o + h_0) - F(x_0 + h_0)) + F(x_0 + h_0) - (F(x_0 + o) - F(x_0) + F(x_0))}{h_0} \\
>&-\epsilon/h_0 + \frac{F(x_0 + h_0) - F(x_0)}{h_0} > \alpha \text{ when taking } \epsilon \text{ small enough}
\end{aligned}$$

so that $x_0 + O$ is an open ball containing x_0 and being contained in $F_{\delta,\alpha}$.

1.2 Problem II

Consider the approximations to the identity $\{K_\delta\}_{\delta>0}$ and take $\delta > 0, x \in \mathbb{R}$ arbitrary. We show $|(f \star K_\delta)(x)| \leq cf^*(x)$ where $f^*(x)$ is the Hardy-Littlewood maximal function, then obviously the result follows.

Now $|\int_{\mathbb{R}} f(x-y)K_\delta(y)dy| \leq \int_{\mathbb{R}} |f(x-y)||K_\delta(y)|dy \leq \frac{A}{\delta}\int_{|y|\leq\delta}|f(x-y)|dy + A\sum_{n=1}^{\infty}\frac{1}{\delta 2^{2n-2}}\int_{\delta 2^{n-1}<|y|\leq\delta 2^n}|f(x-y)|dy \leq \frac{2A}{2\delta}\int_{|y-x|\leq\delta}|f(y)|dy + 2A\sum_{n=1}^{\infty}\frac{1}{2^{n-2}}\frac{1}{2\delta 2^n}\int_{|y-x|\leq\delta 2^n}|f(y)|dy \leq 2Af^*(x) + 4A\sum_{n=0}^{\infty}\frac{1}{2^n}f^*(x) = 10Af^*(x)$, since $|y-x|\leq\delta$ and $|y-x|\leq\delta 2^n$ both describe balls containing y, with measures 2δ and $2\delta 2^n$ in $\mathbb{R}$ respectively, and the supremum in the Hardy-Littlewood maximal function is over all such balls divided by their measure. Also A is independent of both δ and f.

1.3 Problem III

Take $\epsilon > 0$, x and y arbitrary, with the guarantee that $|x-y| < \delta_\epsilon$ for some $\delta_\epsilon > 0$ we must choose. We want to show that under these conditions $|F(x) - F(y)| < \epsilon$. Then without loss of generality assume $x > y$, so that we obtain the equivalent $|\int_y^x f(x)dx| < \epsilon$. We in fact show the stronger result that there exists such a δ_ϵ such that $\int_y^x |f(x)|dx < \epsilon$. Actually that result is immediate as $\lambda([y,x]) = x - y < \delta_\epsilon$ and we can take δ_ϵ as in proposition 1.12(ii) (Stein, p. 65).

1.4 Problem IV

If $g_{[0,1]\times[0,1]}(x,y) = |f(x) - f(y)|$ is assumed integrable on $[0,1]\times[0,1]$ then by Fubini we immediately have that for almost all y, $g^y_{\mathbb{R}}(x) = |f(x) - f(y)|\chi_{[0,1]\times[0,1]}(x,y)$ is integrable. Therefore, for almost all y, $\int_{\mathbb{R}}|f(x) - f(y)|\chi_{[0,1]\times[0,1]}(x,y)dx < \infty \Rightarrow \chi_{[0,1]}(y)\int_{[0,1]}|f(x) - f(y)|dx < \infty$. Pick some $y \in [0,1]$ for which the last equation holds (you can surely do this), then $f(y) = C < \infty$ per assumption. Hence $\int_{[0,1]}|f(x) - C|dx < \infty$, and therefore $\int_{[0,1]}|f(x)|dx \leq \int_{[0,1]}(|f(x) - C| + |C|)dx < \infty$ as $C < \infty$. That is, f is integrable on $[0,1]$.

Question: We do not really need the assumption that f is finite-valued on $[0, 1]$, right?

1.5 Problem V

Since $\lambda_*(E) = \inf \lambda_*(O)$, where the infimum is taken over all open sets $O \supset E$, we in particular have a family of open sets $O_\epsilon \supset E$ such that $\lambda_*(E) > \lambda_*(O_\epsilon) - \epsilon \Rightarrow \lambda_*(E) > \alpha\lambda_*(O_\epsilon)$ with $\alpha = 1 - \frac{\epsilon}{\lambda_*(O_\epsilon)}$. Hence as $\epsilon \to 0$, $\alpha \to 1$, because $\lambda_*(O_\epsilon)$ is bounded from below by $\lambda_*(E) > 0$, and thus for every $0 < \alpha < 1$ there is an ϵ such that for some open set $O = O_\epsilon$ we have $\lambda_*(E) > \alpha\lambda_*(O)$. Now O can be dissected into a countable union of disjoint open intervals, say $O = \bigcup_n^\infty I_n$. I claim that one of these $I = I_n$ in fact satisfies $\lambda_*(E \cap I) > \alpha\lambda_*(I)$. Thus suppose for contradiction that none of these I_n have the desired property. Notice that $\lambda_*(E) = \sum_n^\infty \lambda_*(E \cap I_n)$ and $\lambda_*(O) = \sum_n^\infty \lambda_*(I_n)$. Since per assumption $\forall n, \lambda_*(E \cap I_n) \leq \alpha\lambda_*(I_n)$ we have $\sum_n^\infty \lambda_*(E \cap I_n) \leq \alpha \sum_n^\infty \lambda_*(I_n) \Rightarrow \lambda_*(E) \leq \alpha\lambda_*(O)$. Contradiction, and we are done.