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Assignment # ~~2~~ 3

1 Problems

1.1 Problem I

First note that $f(x-y)g(y)$ is measurable on $\mathbb{R}^2$ by Proposition 3.9 (p. 86), Corollary 3.7 (p. 85) and the fact that products of measurable functions are measurable. So we can apply Tonelli theorem (Theorem 3.2, pp 80-81) to $|f(x-y)g(y)|$ and obtain the chain $\int_{\mathbb{R}^2} |f(x-y)||g(y)|dxdy = \int_{\mathbb{R}}(\int_{\mathbb{R}} |f(x-y)||g(y)|dx)dy = \int_{\mathbb{R}} |g(y)|(\int_{\mathbb{R}} |f(x)|dx)dy = (\int_{\mathbb{R}} |f(x)|dx)(\int_{\mathbb{R}} |g(y)|dy) = ||f||_{L^1} \cdot ||g||_{L^1} < \infty$ by translation invariance and the assumed integrability of both f and g, and therefore we see that $f(x-y)g(y)$ is integrable on $\mathbb{R}^2$. An application of Fubini then shows that $(f \star g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$ too is integrable, in $\mathbb{R}$ that is of course. Also we already had $||f \star g||_{L^1} = \int_{\mathbb{R}} |(f \star g)(x)|dx = \int_{\mathbb{R}} \left|\int_{\mathbb{R}} f(x-y)g(y)dy\right| dx \leq \int_{\mathbb{R}}(\int_{\mathbb{R}} |f(x-y)||g(y)|dy)dx = \int_{\mathbb{R}^2} |f(x-y)||g(y)|dxdy = ||f||_{L^1} \cdot ||g||_{L^1}$ and hence $||f \star g||_{L^1} \leq ||f||_{L^1} \cdot ||g||_{L^1}$. ✓ 10/10

1.2 Problem II

This is just by untangling definitions using Fubini, Problem 1 and the fact that $e^{-2\pi i x t}$ is bounded and measurable $\forall x$: $\widehat{f \star g}(x) = \int_{\mathbb{R}}(\int_{\mathbb{R}} f(t-y)g(y)dy)e^{-2\pi i x t}dt = \int_{\mathbb{R}} g(y)(\int_{\mathbb{R}} f(t-y)e^{-2\pi i x t}dt)dy = \int_{\mathbb{R}} g(y)(\int_{\mathbb{R}} f(t-y)e^{-2\pi i x(t-y)}dt)e^{-2\pi i x y}dy = \int_{\mathbb{R}} g(y)e^{-2\pi i x y}(\int_{\mathbb{R}} f(t)e^{-2\pi i x t}dt)dy = (\int_{\mathbb{R}} g(y)e^{-2\pi i x y}dy)(\int_{\mathbb{R}} f(t)e^{-2\pi i x t}dt) = \hat{f}(x) \cdot \hat{g}(x)$, which is what we wanted. ✓

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Therefore under the supposition that there is an integrable I such that for any $f \in L_1$, $f \star I = f$, we get by considering a collection $f_k = \chi_{[-k,k]}$, $k \geq 1$ of integrable functions that $\forall k$, $\hat{f}_k(x) \cdot \hat{I}(x) = \hat{f}_k(x) \Rightarrow \hat{I}(x) = 1$ on $[-k, k]$, and thus in general $\hat{I}(x) = 1$ on $\mathbb{R}$. Hence for the function I we have the relation $\int_{\mathbb{R}} I(t)e^{-2\pi ixt}dt = 1$, $\forall x \in \mathbb{R}$. But this violates Riemann-Lebesgue lemma if we take $x \to \infty$, thus such an I does not exist.

1.3 Problem III

Let me first show that there is some open (bounded) interval I such that $\int_I |f| > 0$ when f is assumed not identically zero. By definition of integrability having f not identically zero means that there is some positive measure set E such that for any $x \in E$, $f(x) \neq 0$, that is on E, $|f| > 0$. Also this implies that there is some $\delta > 0$ and a positive measure set $F \subset E$ such that for any $x \in F$ we have in fact $|f(x)| > \delta$. To see why this is true consider the collection of measurable sets (since f is assumed integrable and thus measurable) $F_n = \{x \in E : |f(x)| > \frac{1}{n}\}$ defined over $\mathbb{N}$, then $E = \bigcup_n^\infty F_n$. Suppose that $\forall n$ we had $\lambda(F_n) = 0$, then we would have $\lambda(E) = 0$, a contradiction. Hence for some N, $\lambda(F_N) > 0$ and we have $F = F_N$ with $\delta = \frac{1}{N} > 0$. Also then by considering the family $F^n = F \cap [-n, n]$ over $\mathbb{N}$ along the same lines, we find a bounded measurable set $K \subset F$ such that $\lambda(K) > 0$. Taking any bounded open interval $I \supset K$ with $[-1, 1] \subset I$ we obviously have $\int_I |f| > 0$.

Now the main result follows quickly: by construction of our bounded I we have $c = \frac{1}{\lambda(I)} \int_I |f| > 0$ such that for any $x \in Z = I \cap (-1, 1)^c$, $f^*(x) > c \geq \frac{c}{|x|}$, because in the sup we can just take the "ball" I for each of these x. For $x \in Z^c$ we can simply dilate I by a factor $|x| \geq 1$ so that $x \in |x|I$ (having the usual dilation set meaning) and obtain by using $\lambda(|x|I) = |x|\lambda(I)$ that also for those xs $f^*(x) > \frac{1}{|x|\lambda(I)} \int_{|x|I} |f| \geq \frac{1}{|x|} \frac{1}{\lambda(I)} \int_I |f| = \frac{c}{|x|}$.

Therefore f^* is not integrable when $f \neq 0$. This is because $\int_{\{x \in \mathbb{R}: |x| \geq 1\}} \frac{c}{|x|} = \infty$ by elementary calculus for any constant $c > 0$, and the monotonicity property of Lebesgue integrals.

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1.4 Problem IV

E is measurable and therefore there exist $\forall \epsilon > 0$ open sets $O_\epsilon \supset E$ such that $\lambda(O_\epsilon - E) < \epsilon$, and for each of these sets $\lambda(O_\epsilon) = \lambda(O_\epsilon - E \cup E) = \lambda(O_\epsilon - E) + \lambda(E) = \lambda(O_\epsilon - E) < \epsilon$. Let $P_n = O_{1/2^n}$ and define the measurable non-negative functions $f_n = \chi_{P_n}$ and $f = \sum_{n=1}^{\infty} f_n$, then $\int_{\mathbb{R}} f = 1 < \infty$ and thus f is integrable. Pick an arbitrary $x \in E$ and consider an arbitrary sequence of open intervals I_m containing x such that $\lambda(I_m) \to 0$ as $m \to \infty$. Now by the openness of P_n and the fact that $x \in P_n$ by construction, $\forall n$, for any n there exists an N such that whenever $m \geq N$ then $I_m \subset P_n$. In particular then for any l there exists some K such that whenever $k \geq K$, $I_k \subset \bigcap_{i=1}^{l} P_i$. But now consider that by the monotone convergence theorem $\int_{I_m} f(y)dy = \sum_{n=1}^{\infty} \int_{I_m} f_n(y)dy$, and therefore by the above for any l, $\liminf_{m\to\infty} \frac{1}{\lambda(I_m)} \int_{I_m} f(y)dy \geq \liminf_{m\to\infty} \frac{1}{\lambda(I_m)} \sum_{n=1}^{l} \int_{I_m} f_n(y)dy \geq \frac{1}{\lambda(I_k)} \sum_{n=1}^{l} \int_{I_k} f_n(y)dy = l$, and this establishes what we wanted to proof.

you need more constraints on I_m wl for P_n to ensure this

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1.5 Problem V

We look at the set $K = E^c \cap (0,1)$ and prove that $\lambda(K) = 0$, obviously then $\lambda(E) = 1$ and we have established the result. Reducing a little further: by the Lebesgue differentiation theorem (Theorem 1.4) applied to the characteristic function of E there is a measure zero set F such that for any $x \in K \subset E^c$ either x has the property that $\lim_{\lambda(I)\to 0, x\in I} \frac{\lambda(I\cap E)}{\lambda(I)} = 0$ or $x \in F$. I show that it must be the case that $x \in F$ and hence $K \subset F \Rightarrow \lambda(K) = 0$. Taking any arbitrary sequence of intervals $I_k \to 0$ as $k \to \infty$, each containing x and being contained in $[0,1]$, we see that, since $\forall k$, $\frac{\lambda(I_k\cap E)}{\lambda(I_k)} \geq \alpha > 0$, we have a limit $\lim_{k\to\infty} \frac{\lambda(I_k\cap E)}{\lambda(I_k)} > 0$. But this contradicts the property as for any $x \in K \subset (0,1)$ such a limit under the conditions $\lambda(I) \to 0, x \in I$ will eventually have all its intervals contained in $[0,1]$. Thus necessarily $x \in F$.

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