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1. Let A be a measurable set with $\lambda(A) > 0$. Show that $A + A = \{x + y : x, y \in A\}$ contains an open interval.

1. Consider

$$f(y) = \int_{\mathbb{R}} \chi_A(x)\chi_A(y-x)dx.$$

We can take A to be bounded and of finite measure (or just consider sets of the form $A \cap [n, n+1)$). Since $x_A \in L^1$, for any $\varepsilon > 0$ there exists a continuous function with compact support, g, such that $\|\chi_A - g\|_{L^1} < \frac{\varepsilon}{6M}$, where $M > 1$ is an upper bound of g.

$$|f(y+h) - f(y)| = \left| \int_{\mathbb{R}} \chi_A(x)\chi_A(y-x+h) - \chi_A(x)\chi_A(y-x)dx \right|$$

$$\leq \left| \int_{\mathbb{R}} \chi_A(x)\chi_A(y-x+h) - g(x)g(y-x+h)dx \right| + \left| \int_{\mathbb{R}} g(x)g(y-x+h) - \chi_A(x)\chi_A(y-x)dx \right|$$

For the first integral

$$\begin{aligned}
\left| \int_{\mathbb{R}} \chi_A(x)\chi_A(y-x+h) - g(x)g(y-x+h)dx \right| &\leq \int_{\mathbb{R}} |\chi_A(x)\chi_A(y-x+h) - \chi_A(x)g(y-x+h)|\, dx \\
&\quad + \int_{\mathbb{R}} |\chi_A(x)g_A(y-x+h) - g(x)g(y-x+h)|\, dx \\
&\leq \int_{\mathbb{R}} |\chi_A(y-x+h) - g(y-x+h)|\, dx \\
&\quad + M \int_{\mathbb{R}} |\chi_A(x) - g(x)|\, dx \\
&\leq \frac{\varepsilon}{6M} + \frac{\varepsilon}{6} < \frac{\varepsilon}{2}
\end{aligned}$$

For the second integral

$$\begin{aligned}
\left| \int_{\mathbb{R}} g(x)g(y-x+h) - g(x)\chi_A(y-x)dx \right| &\leq \int_{\mathbb{R}} |g(x)g(y-x+h) - g(x)g(y-x)|\, dx \\
&\quad + \int_{\mathbb{R}} |g(x)g_A(y-x) - g(x)\chi_A(y-x)|\, dx \\
&\quad + \int_{\mathbb{R}} |g(x)\chi_A(y-x) - \chi_A(x)\chi_A(y-x)|\, dx \\
&\leq M \int_{\mathbb{R}} |g(y-x+h) - g(y-x)|\, dx \\
&\quad + M \int_{\mathbb{R}} |g_A(y-x) - \chi_A(y-x)|\, dx + \int_{\mathbb{R}} |g(x) - \chi_A(x) \\
&\leq M \int_{\mathbb{R}} |g(y-x+h) - g(y-x)|\, dx + \frac{\varepsilon}{6M} + \frac{\varepsilon}{6}
\end{aligned}$$

Since g is (uniformly) continuous, there exists $1 > \delta > 0$ such that $|h| < \delta$ then implies $|g(y-x+h) - g(y-x)| < \frac{\varepsilon}{6(N+2)M}$, where $[-N, N]$ is the support of g, and

$$M \int_{\mathbb{R}} |g(y-x+h) - g(y-x)|\, dx < \frac{\varepsilon}{6}.$$

For $|h| < \delta$

$$\left| \int_{\mathbb{R}} g(x)g(y-x+h) - g(x)\chi_A(y-x)dx \right| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6M} + \frac{\varepsilon}{6} < \frac{\varepsilon}{2}.$$

Implying for any $\varepsilon > 0$ there exists $\delta > 0$ such that is $|h| < \delta$ then,

$$|f(y+h) - f(y)| < \varepsilon.$$

So f is continuous. Notice by Fubini and translational invariance of the Lebesgue integral that

$$\int_{\mathbb{R}} f(y)dy = \int_{\mathbb{R}} \chi_A(y)dy \int_{\mathbb{R}} \chi_A(x)dx = [\lambda(a)]^2 > 0$$

So $f(z) > 0$ for some $z \in \mathbb{R}$. But since $f(y)$ is continuous there is a neighborhood $(z-\delta, z+\delta)$ over which $f(y) > 0$. Also,

$$\chi_A(x)\chi_A(y-x) = \chi_A(x)\chi_{y-A}(x) = \chi_{A\cap(y-A)}(x).$$

This implies for $y \in (z-\delta, z+\delta)$

$$A \cap (y - A) \neq \emptyset$$

for every such y there exists $a, b \in A$ such that

$$a = y - b \iff a + b = y \in (z-\delta, z+\delta)$$

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2. Let $f : [0,1] \to \mathbb{R}^+$ be measurable. Suppose that there is a universal constant $C > 0$ such that for all integers $k \geq 1$,

$$\int_0^1 f(x)^k dx = C.$$

Prove that there is a measurable set $B \subset [0,1]$ such that $f(X) = \chi_B(x)$ almost everywhere.

2. Consider $E_a = \{x \in [0,1] : f(x) > a\}$. Suppose that $\lambda(E_1) > 0$, then there exists $\alpha > 1$ such that $\lambda(E_\alpha^1) > 0$ (Else $E_1 = \bigcup_{n=1}^{\infty} E_{1+\frac{1}{n}}$ has measure zero). Since $f^n \geq 0$,

$$\int_0^1 [f(x)]^{2n}\, dx \geq \int_{E_\alpha^1} [f(x)]^n\, dx \geq \int_{E_\alpha^1} \alpha^n dx = \lambda(E_\alpha^1)\alpha^n \xrightarrow{n\to\infty} \infty$$

We then must have $0 \leq f(x) \leq 1$ a.e. Consider $E_{(0,1)} = \{x \in [0,1] : 0 < f(x) < 1\}$

$$C = \int_0^1 [f(x)]^n\, dx = \int_{[0,1]\setminus E_{(0,1)}} [f(x)]^n\, dx + \int_{E_{(0,1)}} [f(x)]^n\, dx = \int_{[0,1]\setminus E_{(0,1)}} f(x)dx + \int_{E_{(0,1)}} [f(x)]^n\, dx$$

Because $f(x)^n = f(x)$ a.e. on $[0,1] \setminus E_{(0,1)}$ since $f(x) \in \{0,1\}$ a.e. on $[0,1] \setminus E_{(0,1)}$. We can consider the restriction $f_{E_{(0,1)}} \leq \chi_{E_{(0,1)}}$. Notice $\lim_{n\to\infty} \left[f_{E_{(0,1)}}(x)\right]^n = 0$ and so by the dominated convergence theorem

$$\lim_{n\to\infty} \int_{E_{(0,1)}} [f(x)]^n\, dx = \lim_{n\to\infty} \int_{E_{(0,1)}} \left[f_{E_{(0,1)}}\right]^n dx = 0$$

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We then have

$$C = \lim_{n\to\infty} \int_0^1 [f(x)]^n \, dx = \int_{[0,1]\setminus E_{(0,1)}} f(x)dx + \lim_{n\to\infty} \int_{E_{(0,1)}} [f(x)]^n \, dx = \int_{[0,1]\setminus E_{(0,1)}} f(x)dx$$

And so $\int_{E_{(0,1)}} [f(x)]^n = 0$ for all n. By what was shown in the last assignment $f(x) = 0$ a.e. on $E_{(0,1)}$. But by construction $f(x) > 0$ for $x \in E_{(0,1)}$ implying $\lambda(E_{(0,1)}) = 0$ and $f(x) \in \{0, 1\}$ a.e. on $[0, 1]$ or

$$f(x) = \chi_B(x) \text{ a.e.}$$

for some measurable $B \subset [0, 1]$.

3. Let f be integrable. Prove that there exists a sequence $x_n \to \infty$ such that $x_n |f(x_n)| \to 0$ as $n \to \infty$.

3. By contradiction. Suppose such a sequence does not exist. Then for large enough N, there exists an $\varepsilon_0 > 0$ such that

$$x\,|f(x)| \geq \varepsilon_0 \iff \frac{\varepsilon_0}{x} \leq |f(x)|$$

for all $x > N$. But then

$$\int_N^\infty \frac{\varepsilon_0}{x} dx \leq \int_N^\infty |f(x)|\, dx.$$

The integral on the left diverges while the one on the right is finite, a contradiction.

Justify why

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4. (Riemann-Lebesgue lemma) Let f be integrable, show that,

$$\int_{\mathbb{R}} f(x)\cos(nx)dx \to 0, \qquad \int_{\mathbb{R}} f(x)\sin(nx)dx \to 0$$

as $n \to \infty$. Alternatively (if you're more comfortable with complex exponentials) show that,

$$\int_{\mathbb{R}} f(x)e^{2\pi inx}dx \to 0$$

4. Since the Lebesgue integral is invariant under translations,

$$\int_{\mathbb{R}} f(x)e^{2\pi inx}dx = \int_{\mathbb{R}} f\left(x + \frac{1}{2n}\right) e^{2\pi in\left(x+\frac{1}{2n}\right)}dx = -\int_{\mathbb{R}} f\left(x + \frac{1}{2n}\right) e^{2\pi inx}dx.$$

And so

$$\begin{aligned}
\left|\int_{\mathbb{R}} f(x)e^{2\pi inx}dx\right| &= \left|\frac{1}{2}\int_{\mathbb{R}} \left[f(x) - f\left(x + \frac{1}{2n}\right)\right] e^{2\pi inx}dx\right| \\
&\leq \frac{1}{2}\int_{\mathbb{R}} \left|(x) - f\left(x + \frac{1}{2n}\right)\right| \left|e^{2\pi inx}\right| dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \left|f(x) - f\left(x + \frac{1}{2n}\right)\right| dx \xrightarrow{n\to\infty} 0
\end{aligned}$$

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Which follows from the fact that $f \in L^1(\mathbb{R})$ (See proposition 2.5 Shakarchi and Stein).

5. Prove that given a sequence φ_n and a set of positive measure E, the sequence $\cos(nx + \varphi_n)$ cannot tend to zero and $n \to \infty$, for all $x \in E$

5. Let E be any measurable set with $0 < \lambda(E) < \infty$ and φ_n be any sequence in $\mathbb{R}$. By the Riemann Lebesgue lemma

$$\begin{aligned}
\left|\int_E \cos(nx+\varphi_n)dx\right| &= \left|\int_{\mathbb{R}} \cos(nx+\varphi_n)\chi_E(x)dx\right| \\
&= \left|\int_{\mathbb{R}} (\cos(nx)\cos(\varphi_n) - \sin(nx)\sin(\varphi_n))\chi_E(x)dx\right| \\
&\leq |\cos(\varphi_n)|\left|\int_{\mathbb{R}} \cos(nx)\chi_E(x)dx\right| + |\sin(\varphi_n)|\left|\int_{\mathbb{R}} \sin(nx)\chi_E(x)dx\right| \xrightarrow{n\to\infty} 0
\end{aligned}$$

And so we cannot have $\lim_{n\to\infty} \cos(nx+\varphi_n) = c$ for some $c \in [-1,1] \setminus \{0\}$ (or else by the dominated convergence theorem we would get $\lim_{n\to\infty} \int_E \cos(nx+\varphi_n)dx = c\lambda(E)$.) Also

$$\begin{aligned}
\lim_{n\to\infty} \int_E \sin^2(nx+\varphi_n)dx &= \lim_{n\to\infty} \int_{\mathbb{R}} \sin^2(nx+\varphi_n)\chi_E(x)dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} \left(\frac{1-\cos(2nx+2\varphi_n)}{2}\right)\chi_E(x)dx \\
&= \lim_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}} \chi_E(x)dx + \lim_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}} \cos(2nx+2\varphi_n)\chi_E(x)dx \\
&= \frac{\lambda(E)}{2}
\end{aligned}$$

Suppose that

$$\lim_{n\to\infty} \cos(nx+\varphi_n) = 0$$

for all $x \in E$, then for all $x \in E$

$$\lim_{n\to\infty} \sin^2(nx+\varphi_n) = \lim_{n\to\infty} \left(1-\cos^2(nx+\varphi_n)\right) = 1.$$

By the monotone convergence theorem since $\sin^2(nx+\varphi_n)\chi_E(x) \leq \chi_E$ we would have $\lim_{n\to\infty} \sin^2(nx+\varphi_n)\chi_E(x) = \chi_E(x)$ and

$$\lim_{n\to\infty} \int_E \sin^2(nx+\varphi_n)dx = \lim_{n\to\infty} \int_{\mathbb{R}} \sin^2(nx+\varphi_n)\chi_E(x)dx = \int_{\mathbb{R}} \sin^2(nx+\varphi_n)dx = \lambda(E)$$

contradicting the above.

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