1) \( E_\alpha = \{ x : f(x) > \alpha \} \), \( \alpha > 0 \), let \( E_\alpha = \{ x : f(x) > \alpha \} \). Since \( f \) is measurable, \( f^{-1}(\langle \alpha, \infty \rangle) = \{ x : f(x) > \alpha \} \) is measurable. 

Since for \( x \in E_\alpha \), \( f(x) > \alpha \) and so \( \frac{f(x)}{\alpha} > 1 \). Also, since \( f \equiv 0 \), we get 

\[
\int_{E_\alpha} \frac{f}{\alpha} \, dx = \int_{E_\alpha} 1 \, dx \leq \int_{E_\alpha} 0 \, dx = 0 \leq \int_{E_\alpha} \frac{f}{\alpha} \, dx \leq \int_{E_\alpha} 0 \, dx = 0.
\]

Since \( f \equiv 0 \),

\[
\frac{\int_{E_\alpha} f \, dx}{\alpha} = 1.
\]
2) Let $E_k = \{ x : -f(x) > \frac{k}{2} \}$ for $k \in \mathbb{N}$.

Since $E_k = \{ x : -f(x) > \frac{k}{2} \} \subset \{ x : f(x) < \frac{k}{2} \} = f^{-1}(\mathbb{R}, \frac{k}{2})$ and $f$ is measurable,

$E_k$ is also measurable.

By the previous question,

$\frac{1}{k} \chi_E(E_k) \leq \int_{E_k} f(x) \, dx$.

Since $\forall x \in E_k$, $f(x) < \frac{k}{2} < 0$, $f^-(x) = -\min(f(x), 0) = -f(x)$ \forall $x \in E_k$.

Also, $\forall x \in E_k$, $f(x) < 0$ so $f^+(x) = \max(f(x), 0) = 0 \ \forall x \in E_k$. So $\int_{E_k} f^+ \, dx = 0$

Since $E_k$ is measurable, by assumption $\int_{E_k} f^{-} \, dx \geq 0$

So $\int_{E_k} f^{-} = \int_{E_k} f^+ - \int_{E_k} f^+ = 0 - \int_{E_k} f^+ = 0$.

So $\int_{E_k} f^{-} \leq 0$.

So $\frac{1}{k} \chi_E(E_k) \leq \int_{E_k} f^{-} \, dx$.

So $\int_{E_k} f^{-} \, dx \leq 0$.

So $\frac{1}{k} \chi_E(E_k) \leq k \cdot 0 = 0$. So $\chi_E(E_k) = 0 \ \forall k \in \mathbb{N}$.

Since $E_k \subseteq E_n$, $\chi_E(E_k) = 0$

So $E_n = \{ x : f(x) < 0 \}$ has measure 0.

So $f \geq 0$ almost everywhere.

3) Let $I_{k,n} = \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right]$ \forall $k \in \mathbb{N}$, $n = 0, 1, 2, \ldots$.

Let $f_{k,n}(x) = \chi_{I_{k,n}}(x) = \chi_{[k/2^n, (k+1)/2^n]}(x)$.

Then for a fixed $k$, $\int_{I_{k,n}} f_{k,n} \, dx = \int_{I_{k,n}} 1 \, dx = 1$

So $\chi_{[k/2^n, (k+1)/2^n]}(x)$ is a countable collection of intervals. So we can use a single variable $x$ to denote an $I_{k,n}$.

Then $\lim_{n \to \infty} \int_{I_{k,n}} f_{k,n} \, dx = 0$.

So $\lim_{n \to \infty} \int f \, dx = 0$.
3) Cont'd

So \( \|f_n(x) - 0\|_1 = 1 \)

So for \( \varepsilon = \frac{1}{2} \), there is no \( N \) such that \( \forall n \geq N, \|f_n(x) - f(x)\|_1 < \varepsilon = \frac{1}{2} \).

So \( \lim_{n \to \infty} f_n(x) \neq f(x) \).

So \( 0 < \int_0^1 |f_n(x) - f(x)| \, dx \neq 0 \), get \( \int_0^1 |f_n(x) - f(x)| \, dx \to 0 \)

\( \lambda \) proof.

\[ \text{As } n \to \infty, \int_0^1 |f_n(x) - f(x)| \, dx \to 0, \]

we get

\[ 0 \leq \liminf_{n \to \infty} \lambda(E_{\varepsilon, n}) \leq \lim_{n \to \infty} \int_0^1 |f_n(x) - f(x)| \, dx = 0. \]

So \( \lim_{n \to \infty} \lambda(E_{\varepsilon, n}) = 0. \) So \( \lim_{n \to \infty} \lambda(E_{\varepsilon, n}) = 0. \)

So as \( n \to \infty, \) \( \lambda(\{x : |f_n(x) - f(x)| > \varepsilon\}) \to 0. \)

\[ \square \]

\[ \text{Proof: }\]

Let \( E_{\varepsilon, n} = \{x : |f_n(x) - f(x)| > \varepsilon\}. \) By question 5, \( \forall n \in \mathbb{N}, \)

\[ \int_0^1 \text{I}_{E_{\varepsilon, n}}(x) \, dx = \int_0^1 \lambda(E_{\varepsilon, n}) \, dx. \]

Since \( \lim_{n \to \infty} \lambda(E_{\varepsilon, n}) = 0, \) \( \lim_{n \to \infty} \int_0^1 \lambda(E_{\varepsilon, n}) \, dx = 0. \)

So \( \lim_{n \to \infty} \int_0^1 |f_n(x) - f(x)| \, dx = \int_0^1 \lambda(E_{\varepsilon, n}) \, dx = 0. \)

\[ \square \]
Let $E = \{ x : 0 < x - r_n < 1 \} = \{ x : r_n < x < r_n + 1 \}$

$$\int_{E} f(x-r_n) \, dx = \int_{E} f(x-r_n) \, dx + \int_{E^c} f(x-r_n) \, dx$$

since $f(x-r_n) = 0 \forall x \notin E$.

$$\int_{E} f(x-r_n) \, dx + \int_{E^c} f(x-r_n) \, dx = \int_{E} f(x-r_n) \, dx + 0$$

$$\int_{E} f(x-r_n) \, dx = \int_{E} f(x) \, dx = \int_{0}^{x^2} \, dx = 2 \times x^3 \bigg|_0^1 = 2$$

So $\int_{E} f(x-r_n) \, dx = 2 \forall r_n \in \mathbb{A}$.  

Since $F(x) = \sum_{n=1}^{\infty} f(x-r_n)$,  

$$\frac{\partial}{\partial x} F(x) = \sum_{n=1}^{\infty} \frac{1}{x-r_n} f(x-r_n) \, dx = \sum_{n=1}^{\infty} \frac{1}{x-r_n} f(x-r_n) \, dx$$

So $\int_{E} f(x-r_n) \, dx \sum_{n=1}^{\infty} \frac{1}{x-r_n} f(x-r_n) \, dx = \sum_{n=1}^{\infty} \frac{1}{x-r_n} f(x-r_n) \, dx = \int_{E} f(x-r_n) \, dx = 2 \sum_{n=1}^{\infty} \frac{1}{x-r_n}$

So $\int_{E} f(x-r_n) \, dx = 2 < +\infty$  

Since $F(x) < +\infty$ a.e.  

So $\frac{\partial}{\partial x} F(x) < +\infty$ a.e.,  

let $I$ be an interval with endpoints $a, b$.  

Then for $x \in I$ and $r_k \in A$,  

$$\frac{1}{x-r_k} f(x-r_k) \approx \frac{1}{r_k^2} f(x-r_k) \approx \frac{1}{r_k^2} f(x-r_k)$$

Now let $I_n = [n, n+1]$.  

Then  

$$\frac{1}{x-r_k} f(x-r_k) = \frac{1}{x-r_k} \sum_{n=1}^{\infty} \frac{1}{x-r_k} f(x-r_k) = \frac{1}{x-r_k} \sum_{n=1}^{\infty} \frac{1}{x-r_k} f(x-r_k)$$

So $f(x-r_k) = \frac{1}{x-r_k} \sum_{n=1}^{\infty} \frac{1}{x-r_k} f(x-r_k) \approx \frac{1}{x-r_k} \sum_{n=1}^{\infty} \frac{1}{x-r_k} f(x-r_k) = \frac{1}{x-r_k} \sum_{n=1}^{\infty} \frac{1}{x-r_k} f(x-r_k) = \frac{1}{x-r_k} \sum_{n=1}^{\infty} \frac{1}{x-r_k} f(x-r_k)$
5) Let $E_a = \{x : 1/a \leq x \leq a^3\}$ and if be integrable.

$$\int_{E_a} \chi_{[0,1/(a^3)]} \, dx = \int \left( \int_{[0,1/(a^3)]} \chi_{[0,1/(a^3)]} \, dx \right) \, da$$

Since $\forall x < 0$, $\alpha \in [0,1/(a^3)]$ and so $\frac{1}{1/(a^3)}(x) = 0 \forall x < 0$.

$$= \int \left( \int_{[0,1/(a^3)]} \chi_{[0,1/(a^3)]} \, dx \right) \, da$$

Since $\forall x \in E_a$, $1/(a^3) > \alpha$ and $\forall x \in E_a$, $1/(a^3) \leq \alpha$,

$$\frac{1}{1/(a^3)}(x) = 0 \forall x \in E_a.$$ 

So

$$\int_{E_a} \left( \int_{[0,1/(a^3)]} \chi_{[0,1/(a^3)]} \, dx \right) \, da = \int \frac{1}{1/(a^3)} \, da \int_{E_a} \chi_{[0,1/(a^3)]} \, dx = \int 0 \, da \int_{E_a} \chi_{[0,1/(a^3)]} \, dx.$$

Since $\forall x \in E_a$, $1/(a^3) > \alpha$ and so

$$\frac{1}{1/(a^3)}(x) = 1 \forall x \in E_a$$

So

$$= \int_{E_a} \chi_{[0,1/(a^3)]} \, dx = \int \chi_{(E_a)} \, da$$

But the Fubini theorem in the textbook.

$$= \int \chi_{E_a} \, dx$$

So

$$\int_{E_a} \chi_{[0,1/(a^3)]} \, dx = \int \chi_{E_a} \, dx$$