

Assignment 2

Problem 1. (1) We prove that a closed set is a G_δ set. Let \mathcal{D} be a closed set. We set

$$\mathcal{O}_n = \{x : d(x, \mathcal{D}) < \frac{1}{n}\} \quad (1)$$

where $d(x, \mathcal{D}) = \inf\{|x - y| : y \in \mathcal{D}\}$. And we can write $\mathcal{D} = \bigcap_{n=1}^{\infty} \mathcal{O}_n$. Here we prove \mathcal{O}_n is an open set. Let $x \in \mathcal{O}_n$. Set $r' = \frac{1}{n} - d(x, \mathcal{D})$. We consider the open ball $B(x, r')$. Let $z \in B(x, r')$. $d(z, \mathcal{D}) \leq d(z, x) + d(x, \mathcal{D}) < r' + d(x, \mathcal{D}) = \frac{1}{n} - d(x, \mathcal{D}) + d(x, \mathcal{D}) = \frac{1}{n}$. So $B(x, r') \subset \mathcal{O}_n \implies \mathcal{O}_n$ is open.

Then we prove that a open set is F_δ set. \mathcal{D} is closed, then \mathcal{D}^c is open.

$$\mathcal{D}^c = \left(\bigcap_{n=1}^{\infty} \mathcal{O}_n\right)^c = \bigcup_{n=1}^{\infty} \mathcal{O}_n^c \quad (2)$$

\mathcal{O}_n is open so \mathcal{O}_n^c is closed. Thus, an open set is a F_δ set.

We have the set of rational numbers \mathbb{Q} , which is a F_δ set but not a G_δ set. Since every singleton in \mathbb{Q} is closed and \mathbb{Q} is the countable union of singletons, which implies that \mathbb{Q} is a F_δ set. Now, suppose that \mathbb{Q} is the countable intersection of open sets. i.e. $\mathbb{Q} = \bigcap_{n=1}^k \mathcal{O}_n$ (where $1 \leq k \leq \infty$). Pick $x \in \bigcap_{n=1}^k \mathcal{O}_n$. Then there must exist a $r > 0$ such that $B(x, r) \subset \bigcap_{n=1}^k \mathcal{O}_n$. In other words, $B(x, r) \subset \mathbb{Q}$. By density of irrational numbers, we can always find some irrational number $\alpha \in B(x, r)$, which implies that $B(x, r) \not\subset \mathbb{Q}$. Contradiction. Thus, \mathbb{Q} is a F_δ set but not a G_δ set.

Problem 2. First, for each n , we pick the sequence c_n such that

$$\lambda(\{x : \left|\frac{f_n(x)}{c_n}\right| > \frac{1}{n}\}) < 2^{-n} \quad (3)$$

And we denote

$$E_n = \{x : \left|\frac{f_n(x)}{c_n}\right| > \frac{1}{n}\} \quad (4)$$

By (3), we have that $\sum_{n=1}^{\infty} \lambda(E_n) < \infty$. We set $E = \{x : x \in E_n \text{ for infinitely many } n\}$. By **Borel-Cantelli lemma**, we have that $\lambda(E) = 0$. This implies that c_n is a sequence such that $\left|\frac{f_n(x)}{c_n}\right| \leq \frac{1}{n}$ for almost everywhere in $[0, 1]$. And then we let $n \rightarrow \infty$. So

$$\lim_{n \rightarrow \infty} \frac{f_n(x)}{c_n} = 0 \text{ for almost everywhere } [0, 1]$$

use continuity and specify at which point
this looks like uniform continuity

6/110
Problem 3. We suppose that $f(x) = \mathbf{1}_{[0,1]}(x)$ almost everywhere. We look at the interval $(-\frac{\delta}{2}, 0) \cup (0, \frac{\delta}{2})$. Since $f(x) = \mathbf{1}_{[0,1]}(x)$ almost everywhere, we can always find two points $x_0 \in (-\frac{\delta}{2}, 0)$ and $y_0 \in (0, \frac{\delta}{2})$. Note that we also have $f(x_0) = 0$ and $f(y_0) = 1$. Now, we take $\varepsilon = \frac{1}{2}$ and let $y_0 \in (0, \frac{\delta}{2})$ be given. But for $x_0 \in (-\frac{\delta}{2}, 0)$, $|f(x_0) - f(y_0)| = 1 > \varepsilon$. Thus, this $f(x)$ can not be continuous everywhere.

Problem 4. Let \mathcal{D} be an arbitrary open disc in \mathbb{R}^2 . And we denote the open rectangle $\mathcal{R}_i = (a_i, b_i) \times (c_i, d_i)$ with $\mathcal{R}_i \subset \mathcal{D}$ for all i . And we claim that the boundary $\partial\mathcal{R}_i$ does not belong to any rectangles but it is in the open disc \mathcal{D} .

✓
10
Proof of the Claim. We pick a point $x \in \partial\mathcal{R}_i$. First, $x \in \mathcal{D}$. Suppose that $x \in \mathcal{R}_j$ for some $j \neq i$. Since \mathcal{R}_j is open, we can always find an open ball $B(x, r) \subset \mathcal{R}_j$. But $x \in \partial\mathcal{R}_i$, which means $B(x, r) \cap \mathcal{R}_i \neq \emptyset$. This is clearly a contradiction since \mathcal{R}_i and \mathcal{R}_j are disjoint. ($B(x, r) \subset \mathcal{R}_j \implies B(x, r) \cap \mathcal{R}_i = \emptyset$). \square

By the proof of the claim we can deduce that an open disc in \mathbb{R}^2 is not a disjoint union of open rectangles.

Problem 5. (1) First, we have $f(0) = f(0+0) = f(0) + f(0) = 2f(0) \implies f(0) = 0$ and $f(x + (-x)) = f(x) + f(-x) = 0 \implies f(x) = -f(-x)$.

Claim A: f is continuous at 0.

Proof of the Claim A. We prove by contradiction. Suppose $f(x)$ is not continuous at 0. We fix ε_0 . Then for any $\delta > 0$, there exists $|x| < \delta$ such that $|f(x)| \geq \varepsilon_0$. ($f(0) = 0$) Now, since f is Lebesgue measurable and finite value on $(-k, k)$ ($k \in \mathbb{R}$), we apply the Lusin Theorem and let $\varepsilon_k \rightarrow 0$. There must exist a closed set $F_{\varepsilon_k} \subset (-k, k)$ with $\lambda(F_{\varepsilon_k}) = 2k$ such that $f|_{F_{\varepsilon_k}}$ is continuous. Also f is uniformly continuous on F_{ε_k} . Now, since f is uniformly continuous on F_{ε_k} , we fix such $0 < \varepsilon < 2\varepsilon_0 - f(0)$, then we can find a δ_ε such that $\forall x, y \in F_{\varepsilon_k}$ such that $|x - y| < \delta_\varepsilon \implies |f(x) - f(y)| < \varepsilon$. Let $\delta = \frac{\delta_\varepsilon}{2}$. There must exist a $x_0 \in (-\frac{\delta_\varepsilon}{2}, \frac{\delta_\varepsilon}{2})$ such that $|f(x_0)| \geq \varepsilon_0$. Without loss of generality, we have $f(x_0) \geq 0$. So we have $f(x_0) \geq \varepsilon_0$. $f(-x_0) = f(0) - f(x_0) \leq f(0) - \varepsilon_0$. This implies $|f(x_0) - f(-x_0)| \geq 2\varepsilon_0 - f(0) > \varepsilon$. But $f(x)$ is uniformly continuous on F_{ε_k} . Contradiction.

✓
Remark: Here for safety I keep $f(0)$ since I don't know whether 0 is in F_{ε_k} or not. But it doesn't matter since we can have the freedom to choose this $f(0) < 2\varepsilon_0$ to make all the proof works. But you might ask how could we guarantee the existence of $x_0 \in F_{\varepsilon_k}$. We can pair the $(x_0, -x_0)$. If only one of them exists then we clearly can't choose this x_0 , but if there are no x_0 and $-x_0$ that both exists, we can't have this Lebesgue measure equals to $2k$, it has to be k . That's why I let $\varepsilon_k \rightarrow 0$. \square

Now we prove f is continuous on \mathbb{R} .

Proof. Let $c \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. Since f is continuous at 0, $\exists \delta > 0$ such that $|x| < \delta \implies |f(x)| < \varepsilon$. If $|x - c| < \delta$, then $|f(x) - f(c)| = |f(x) + f(-c)| = |f(x - c)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that f is continuous at c . And since $c \in \mathbb{R}$ is arbitrary, we conclude that f is continuous on \mathbb{R} . \square

(2) We prove that $f(x) = xf(1)$.

Proof. By induction we can easily show that $f(x_1 + x_2 + \dots + x_k) = f(x_1) + f(x_2) + \dots + f(x_k)$. Let $m \in \mathbb{N}$. Then $f(1) = f(\sum_{i=1}^m (1/m)) = \sum_{i=1}^m f(1/m) = mf(1/m)$ and we deduce that $f(1/m) = (1/m)f(1)$. Let $n \in \mathbb{N}$, then $f(n/m) = f(\sum_{i=1}^n (1/m)) = \sum_{i=1}^n f(1/m) = nf(1/m) = (n/m)f(1)$. Since $(n, m) \in \mathbb{N}^2$ so we can deduce that $f(q) = qf(1)$ for all $q \in \mathbb{Q}_{>0}$. Since $f(x) = -f(-x)$ for all x , we can conclude that $f(r) = rf(1)$ for all $r \in \mathbb{Q}$.

Let now $c \in \mathbb{R}$ be arbitrary. By density of rational numbers, we can find a sequence (r_n) such that $r_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} r_n = c$. Since f is continuous at c , by the sequential criterion, $f(c) = \lim_{n \rightarrow \infty} f(r_n) = \lim_{n \rightarrow \infty} r_n f(1) = cf(1)$. Thus, we conclude that $f(x) = xf(1)$ for all $x \in \mathbb{R}$. □

✓
515