THE RIEMANN ZETA FUNCTION ON VERTICAL ARITHMETIC PROGRESSIONS

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Abstract. We show that the twisted second moments of the Riemann zeta function averaged over the arithmetic progression $\frac{1}{2} + i(an + b)$ with $a > 0$, b real, exhibits a remarkable correspondance with the analogous continuous average and derive several consequences. For example, motivated by the linear independence conjecture, we show at least one third of the elements in the arithmetic progression $an + b$ are not the ordinates of some zero of $\zeta(s)$ lying on the critical line. This improves on earlier work of Martin and Ng. We then complement this result by producing large and small values of $\zeta(s)$ on arithmetic progressions which are of the same quality as the best Ω results currently known for $\zeta(\frac{1}{2} + it)$ with t real.

1. Introduction

In this paper, we study the behavior of the Riemann zeta function $\zeta(s)$ in vertical arithmetic progressions on the critical line. To be more precise, fix real numbers $\alpha > 0$ and β. We are interested in the distribution of values of $\zeta(1/2 + i(\alpha\ell + \beta))$ as ℓ ranges over the integers in some large dyadic interval $[T, 2T]$. Here are some specific questions of interest:

(1) How does the mean square $\sum_{\ell\in[T,2T]} |\zeta(\frac{1}{2} + i\ell)|^2$ compare to $\int_T^{2T} |\zeta(\frac{1}{2} + it)|^2 dt$?
(2) Does the mean square of $\zeta(s)$ distinguish arithmetic sequences? That is, does $\sum_{\ell\in[T,2T]} |\zeta(1/2 + i(\alpha\ell + \beta))|^2$ depend on α and β?
(3) What about the case $\sum_{\ell\in[T,2T]} |\zeta(\frac{1}{2} + i(\alpha\ell + \beta))B(\frac{1}{2} + i(\alpha\ell + \beta))|^2$, where $B(s)$ is an arbitary Dirichlet polynomial? In the special when $B(s)$ is a mollifier, the continuous average of $\zeta(\frac{1}{2} + it)B(\frac{1}{2} + it)$ has been shown to be close to 1. Does $B(s)$ still act the same way when restricted to the discrete sequence $\frac{1}{2} + i(\alpha\ell + \beta)$?

For most - but not all - values of α and β our results suggest that the average behavior of $\zeta(\frac{1}{2} + i(\alpha\ell + \beta))$ is similar to that of a unitary family such as $L(\frac{1}{2}; \chi)$.

Besides being of independent interest the above three questions are motivated by the linear independence conjecture, which we approach through two simpler questions:

(1) Can $\zeta(s)$ vanish at many (or most) of the points $\frac{1}{2} + i(\alpha\ell + \beta)$?
(2) Can $\zeta(s)$ be extremely large or small at a point of the form $\frac{1}{2} + i(\alpha\ell + \beta)$? Are the extreme values at $\frac{1}{2} + i(\alpha\ell + \beta)$ comparable to those of $\zeta(\frac{1}{2} + it)$ with $t \in [T; 2T]$?

We begin with some mean square results.

1.1. **Mean value estimates.** The distribution of values of $\zeta(s)$ on the critical line has been studied extensively by numerous authors and in particular the moments of $\zeta(s)$ have received much attention. Here, much effort has gone into the study of the continuous moments both on the critical line and to the right of the critical line. Generally the study of such moments is easier to conduct to the right of the critical line, and it is the critical line which holds the most interest. For discrete averages of the type we consider, the

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fourth moment of $\zeta(s)$ has been studied to the right of the critical line by A. Good [3]. Consider a Dirichlet polynomial $B(s)$ with,

$$B(s) = \sum_{n \le T^\theta} \frac{b(n)}{n^s}, \text{ and } b(n) \ll d_A(n) \tag{1}$$

for some fixed, but arbitrary $A > 0$. Throughout we will assume that the coefficients $b(n)$ are real.

Theorem 1. *Let $B(s)$ be as above. Let $\phi(\cdot)$ be a smooth compactly supported function, with support in $[1, 2]$. If $\theta < \frac{1}{2}$, then as $T \to \infty$.*

$$\sum_\ell |\zeta(\tfrac{1}{2}+i\ell)B(\tfrac{1}{2}+i\ell)|^2 \cdot \phi\left(\frac{\ell}{T}\right) = \int_{\mathbb{R}} |\zeta(\tfrac{1}{2}+it)B(\tfrac{1}{2}+it)|^2 \cdot \phi\left(\frac{t}{T}\right) dt + O_A(T(\log T)^{-A})).$$

Since $\zeta(s)B(s)$ oscillates on a scale of $2\pi/\log T$ it is interesting that we can reconstruct accurately the continuous average of $\zeta(s)B(s)$ only by sampling at the integers. The reader may be amused by examining the same statement for $\sin x$ or $\sin(\log(|x|+1)x)$, which will be equivalent to the equidistribution of certain sequences modulo 1.

Theorem 1 depends on the fact that we are summing over the integers, and specifically on the fact that the sequence $e^{2\pi\ell}$ cannot be well approximated by rational numbers. To amplify this dependence, let us consider the second moment of $\zeta(s)$ averaged over an arithmetic progression $\alpha n + \beta$, with arbitrary $\alpha > 0$ and β. In this context, our result will depend on the diophantine properties of $e^{2\pi\ell/\alpha}$. Let

$$\delta(\alpha, \beta) = \begin{cases} 0 & \text{if } e^{2\pi\ell/\alpha} \text{ is irrational for all } \ell > 0 \\ \frac{2\cos(\beta\log(m/n))\sqrt{mn}-2}{mn+1-2\sqrt{mn}\cos(\beta\log(m/n))} & \text{if } e^{2\pi\ell/\alpha} \text{ is rational for some } \ell > 0 \end{cases}$$

with $m/n \neq 1$ denoting the smallest reduced fraction having a representation in the form $e^{2\pi\ell/\alpha}$ for some $\ell > 0$. Then we have the following asymptotic result for the second moment of the Riemann zeta function.

Theorem 2. *Let $\phi(\cdot)$ be a smooth compactly supported function, with support in $[1, 2]$. Let $\alpha > 0$, β be real numbers. Then, as $T \to \infty$,*

$$\sum_\ell |\zeta(\tfrac{1}{2}+i(\alpha\ell+\beta))|^2 \cdot \phi\left(\frac{\ell}{T}\right) = \int_{\mathbb{R}} |\zeta(\tfrac{1}{2}+i(\alpha t+\beta))|^2 \cdot \phi\left(\frac{t}{T}\right) dt \cdot (1+\delta(\alpha,\beta)+o(1))$$

In the above, $o(1)$ denotes a quantity tending to 0 as T grows, which depends on the diophantine properties of α and β. Our methods allow us to prove an analogous result for the second moment of $\zeta(s)$ twisted by a Dirichlet polynomial over an arbitrary vertical arithmetic progression and also to recover Good's estimate [3] for the fourth moment of the Riemann zeta-function off the half-line. See Propositions 1 and 2 for more details.

In contrast to Theorem 2, the dependence on the diophantine properties of α and β is nullified when B is a mollifier. To be precise, let $\phi(\cdot)$ be a smooth compactly supported function, with support in $[1, 2]$, and define

$$M_\theta(s) := \sum_{n \le T^\theta} \frac{\mu(n)}{n^s} \cdot \left(1 - \frac{\log n}{\log T^\theta}\right).$$

Then we have the following Theorem.

Theorem 3. *Let the mollified second moment be defined as*

$$\mathcal{J} := \sum_\ell |\zeta(\tfrac{1}{2}+i(\alpha\ell+\beta))M_\theta(\tfrac{1}{2}+i(\alpha\ell+\beta))|^2 \phi\left(\frac{\ell}{T}\right). \tag{2}$$

Let $0 < \theta < \frac{1}{2}$ and $a > 0$ and b be real numbers. Then,

$$\mathcal{J} = \int_{\mathbb{R}} \left|(\zeta \cdot M_\theta)(\tfrac{1}{2} + i(\alpha t + \beta))\right|^2 \cdot \phi\left(\frac{t}{T}\right) dt + O\left(\frac{T}{(\log T)^{1-\varepsilon}}\right)$$

The lack of dependence on the diophantine properties of α and β in Theorem 3 gives the non-vanishing proportion of $\frac{1}{3}$ in Theorem 4 below.

1.2. **Non-vanishing results.** One of the fundamental problems in analytic number theory is determination of the location of the zeros of L-functions. Here, one deep conjecture about the vertical distribution of zeros of $\zeta(s)$ is the Linear Independence Conjecture (LI), which states that the ordinates of non-trivial zeros of $\zeta(s)$ are linearly independent over $\mathbb{Q}$. In general, it is believed that the zeros of L-functions do not satisfy any algebraic relations, but rather appear to be "random" transcendental numbers. Classically, Ingham [4] linked the linear independence conjecture for the Riemann zeta-function with the oscilations of $M(x) = \sum_{n \leq x} \mu(n)$, in particular offering a conditional disproof of Merten's conjecture that $|M(x)| \leq \sqrt{x}$ for all x large enough. There are a number of connections between LI and the distribution of primes. For instance, Rubinstein and Sarnak [9] showed a connection between LI for Dirichlet L-functions and prime number races, and this has appeared in the work of many subsequent authors.

LI appears to be far out of reach of current technology. However, it implies easier conjectures which may be more tractable. One of these is that the vertical ordinates of nontrivial zeros of $\zeta(s)$ should not lie in an arithmetic progression. To be more precise, for fixed $\alpha > 0$, $\beta \in \mathbb{R}$, let

$$P_{\alpha,\beta}(T) = \frac{1}{T} \cdot \mathrm{Card}\{T \leq \ell \leq 2T : \zeta(\tfrac{1}{2} + i(\alpha\ell + \beta)) \neq 0\}.$$

Then what kind of lower bounds can we prove for $P_{\alpha,\beta}(T)$ for large T? Recently, improving on the work of numerous earlier authors, Martin and Ng [8] showed that $P_{\alpha,\beta}(T) \gg_{\alpha,\beta} (\log T)^{-1}$ which misses the truth by a factor of $\log T$. In this paper, we prove the following improvement.

Theorem 4. *Let $\alpha > 0$ and β be real. Then, as $T \to \infty$,*

$$P_{\alpha,\beta}(T) \geq \frac{1}{3} + o(1).$$

The proof of Theorem 4 leads easily to the result below.

Corollary 1. *Let $\alpha > 0$ and β be real. Then, as $T \to \infty$,*

$$|\zeta(\tfrac{1}{2} + i(\alpha\ell + \beta)| \geq \varepsilon(\log \ell)^{-1/2}$$

for more than $(\frac{1}{3} - C\varepsilon)T$ integers $T \leq \ell \leq 2T$, with C an absolute constant.

Theorem 4 is proven by understanding both a mollifed discrete second moment (see Theorem 3) and a mollified discrete first moment. Our methods extend without modification to prove the analogous result for Dirichlet L-functions. The constants $\frac{1}{3}$ represents the limits of the current technology - see for example [5] for the case of non-vanishing of Dirichlet L-functions at the critical point.

Of course, we expect that $P_{\alpha,\beta}(T) = 1 + O(T^{-1})$. Assuming the Riemann Hypothesis (RH), Ford, Soundararajan and Zaharescu [2] showed $P_{\alpha,\beta}(T) \geq \frac{1}{2} + o(1)$ as $T \to \infty$. Assuming RH and Montgomery's Pair Correlation Conjecture they showed [2] that $P_{\alpha,\beta}(T) \geq 1 - o(1)$ as $T \to \infty$. Assuming a very strong hypothesis on the distribution of primes in short intervals, it is possible to show that $P_{\alpha,\beta}(T) = 1 - O(T^{-\delta})$ for some $\delta > 0$.

Note that the rigid structure of the arithmetic progression is important. Since there is a zero of $\zeta(s)$ in every interval of size essentially $(\log\log\log T)^{-1}$ in $[T, 2T]$ (see [7]) minor perturbations of the arithmetic progression renders our result false.

1.3. **Large and small values.** We now complement Theorem 4 by exhibiting large and small values of $\zeta(s)$ at discrete points $\frac{1}{2} + i(\alpha\ell + \beta)$ using Soundararajan's resonance method [10]. Previously A. Good pointed out in [3] that his result imply that $\zeta(\sigma + i(\alpha\ell + \beta)) = \Omega(1)$ for fixed $\sigma > \frac{1}{2}$ and infinitely many ℓ's.

Theorem 5. *Let $\alpha > 0$ and β be real. Then, for infinitely many $\ell > 0$,*

$$|\zeta(\tfrac{1}{2} + i(\alpha\ell + \beta))| \gg \exp\left((1 + o(1))\sqrt{\frac{\log \ell}{6\log\log \ell}}\right)$$

and for infinitely many ℓ,

$$|\zeta(\tfrac{1}{2} + i(\alpha\ell + \beta))| \ll \exp\left(-(1 + o(1))\sqrt{\frac{\log \ell}{6\log\log \ell}}\right).$$

The $o(1)$ in this result is independent of the diophantine properties of α and β. Since we expect $\zeta(\frac{1}{2} + i(\alpha\ell + \beta)) \neq 0$ for essentially all ℓ, it is interesting to produce values of ℓ at which $\zeta(\frac{1}{2} + i(\alpha\ell + \beta))$ is extremely small. Furthermore, the large values of $\zeta(\frac{1}{2} + i(\alpha\ell + \beta))$ over a discrete set of points above are almost of the same quality as the best results for large values of $\zeta(\frac{1}{2} + it)$ with t real. In the latter case, the best result is due to Soundararajan [10]. We have not tried to optimize in Theorem 5 and perhaps the same methods might lead to the constant 1 rather than $1/\sqrt{6}$.

1.4. **Technical propositions.** The proofs of our Theorems rests on a technical Proposition, and its variant, which may be of independent interest. With $B(s)$ defined as in (1), consider the difference between the discrete average and the continuous average,

$$\mathcal{E} := \sum_{\ell} |(\zeta \cdot B)(\tfrac{1}{2} + i(\alpha\ell + \beta))|^2 \phi\left(\frac{\ell}{T}\right) - \int_{\mathbb{R}} |(\zeta \cdot B)(\tfrac{1}{2} + i(\alpha t + \beta))|^2 \phi\left(\frac{t}{T}\right) dt.$$

Proposition 1 below shows that understanding $\mathcal{E}$ boils down to understanding the behavior of sums of the form

$$F(a_\ell, b_\ell, t) := \sum_{r \geqslant 1} \frac{1}{r} \sum_{h,k \leqslant T^\theta} b(k)b(h) \sum_{\substack{m,n \geqslant 1 \\ mk = a_\ell r \\ nh = b_\ell r}} W\left(\frac{2\pi mn}{\alpha t + \beta}\right) \tag{3}$$

where $W(x)$ is a smooth function defined as

$$W(x) := \frac{1}{2\pi i} \int_{(\varepsilon)} x^{-w} \cdot G(w) \frac{dw}{w}$$

with $G(w)$ an entire function of rapid decay along vertical lines $G(x + iy) \ll_{x,A} |y|^{-A}$, such that $G(w) = G(-w)$, $G(0) = 1$, and satisfying $G(\bar{w}) = \overline{G(w)}$ (to make $W(x)$ real valued for x real). For example we can take $G(w) = e^{w^2}$. Notice that $W(x) \ll 1$ for $x \leq 1$ and $W(x) \ll_A x^{-A}$ for $x > 1$.

Of course, the expression in (3) should not depend on the choice of W. In fact, $F(a_l, b_l, t)$ can also be written as

$$\sum_{m,n \leq T^\theta} \frac{b(m)b(n)}{mn} \cdot (ma_\ell, nb_\ell) \cdot \mathcal{H}\left((\alpha t + \beta) \cdot \frac{(ma_\ell, nb_\ell)^2}{2\pi m a_\ell n b_\ell}\right) \tag{4}$$

where $\mathcal{H}(x)$ is a smooth function such that,

$$\mathcal{H}(x) = \begin{cases} \frac{1}{2} \cdot \log x + \gamma + O_A(x^{-A}) & \text{if } x \gg 1 \\ O_A(x^A) & \text{if } x \ll 1 \end{cases}$$

As seen in a theorem of Balasubramanian, Conrey and Heath-Brown [1] the continuous t average over $T \leq t \leq 2T$ of $|\zeta(\frac{1}{2}+it)B(\frac{1}{2}+it)|^2$ gives rise to (4) with $a_\ell = 1 = b_\ell$. For technical reasons it is more convenient for us to work with the smooth version (3).

Proposition 1. *Let $0 < \theta < 1/2$. For each $\ell > 0$, let (a_ℓ, b_ℓ) denote (if it exists) the unique tuple of co-prime integers such that $a_\ell b_\ell > 1$, $b_\ell < T^{1/2-\varepsilon}e^{-\pi\ell/\alpha}$ and*

$$\left| \frac{a_\ell}{b_\ell} - e^{2\pi\ell/\alpha} \right| \leq \frac{e^{2\pi\ell/\alpha}}{T^{1-\epsilon}}. \tag{5}$$

If such a pair (a_ℓ, b_ℓ) exists, then let

$$H(\ell) = \frac{(a_\ell/b_\ell)^{i\beta}}{\sqrt{a_\ell b_\ell}} \int_{-\infty}^{\infty} \phi\Big(\frac{t}{T}\Big) \cdot \exp\Big(- 2\pi i t\Big(\frac{\alpha \log \frac{a_\ell}{b_\ell}}{2\pi} - \ell\Big)\Big) \cdot F(a_\ell, b_\ell, t) dt,$$

and otherwise set $H(\ell) = 0$. Then,

$$\mathcal{E} = 4\mathrm{Re} \sum_{\ell > 0} H(\ell) + O(T^{1-\varepsilon}).$$

More generally we can consider

$$\mathcal{E}' = \sum_{\ell} |B(\tfrac{1}{2} + i(\alpha\ell + \beta))|^2 \phi\Big(\frac{\ell}{T}\Big) - \int_{\mathbb{R}} |B(\tfrac{1}{2} + i(\alpha t + \beta))|^2 \phi\Big(\frac{t}{T}\Big) dt.$$

In this case our results depend on

$$F'(a_\ell, b_\ell) := \sum_{r \geq 1} \frac{b(a_\ell r) b(b_\ell r)}{r}$$

where we adopted the convention that $b(n) = 0$ for $n > T^\theta$. Then the analogue of Proposition 1 is stated below.

Proposition 2. *Let $0 < \theta < 1$. For each $\ell > 0$ let (a_ℓ, b_ℓ) denote (if it exists) the unique tuple of co-prime integers such that $a_\ell b_\ell > 1$, $b_\ell < T^{1/2-\varepsilon}e^{-\pi\ell/\alpha}$ and*

$$\left| \frac{a_\ell}{b_\ell} - e^{2\pi\ell/\alpha} \right| \leq \frac{e^{2\pi\ell/\alpha}}{T^{1-\epsilon}}. \tag{6}$$

Then,

$$\mathcal{E}' = 2\Re \sum_{\ell > 0} \frac{(a_\ell/b_\ell)^{i\beta}}{\sqrt{a_\ell b_\ell}} \cdot \hat{\phi}\Big(\frac{\alpha \log \frac{a_\ell}{b_\ell}}{2\pi} - \ell \Big) F'(a_\ell, b_\ell) + O(T^{1-\varepsilon})$$

where in the summation over ℓ we omit the terms for which the pair (a_ℓ, b_ℓ) does not exist.

The proof of Proposition 2 is very similar (in fact easier!) than that of Proposition 1, and for this reason we omit it. To recover Good's result [3] for the fourth moment of the Riemann zeta-function off the half-line, we only need Proposition 2 for a Dirichlet polynomial of length T^ε.

One can ask about the typical distribution of $\log \zeta(\frac{1}{2} + i(\alpha\ell + \beta))$. This question is out of reach if we focus on the real part of $\log \zeta(s)$ since we cannot even guarantee that almost all $\frac{1}{2} + i(\alpha\ell + \beta)$ are not zeros of the Riemann zeta-function. On the Riemann

Hypothesis, using Proposition 2 and Selberg's methods, one can prove a central limit theorem for $S(\alpha\ell+\beta)$ with $T \le \ell \le 2T$. We will not pursue this application here.

We deduce Theorems 1 and 2 from Proposition 1 in Section 2. We then prove Theorem 3 in Section 3, complete the proof of Theorem 4 in Section 4, and prove Theorem 5 in Section 5. Finally, we prove Proposition 1 in Section 6.

2. Proof of Theorems 1 and 2

Proof of Theorem 1. Set $\alpha = 1$ and $\beta = 0$. By Proposition 1 it is enough to show that $\mathcal{E} \ll T(\log T)^{-A}$. Since $W(x) \ll x^{-A}$ for $x > 1$ and $W(x) \ll 1$ for $x \le 1$, we have, for $T \le t \le 2T$

$$F(a_\ell, b_\ell, t) \ll 1 + \sum_{r \ge 1} \frac{1}{r} \sum_{h,k \le T^\theta} |b(k)b(h)| \sum_{\substack{m,n \le T^{1+\varepsilon} \\ mk = a_\ell r \\ nh = b_\ell r}} 1 \ll \sum_{r \le T^2} \frac{c(a_\ell r)c(b_\ell r)}{r} + 1$$

where $c(n) := \sum_{d|n} |b(d)| \ll d_{A+1}(n)$. Therefore $F(a_\ell, b_\ell, t) \ll (a_\ell b_\ell)^\varepsilon T(\log T)^B$. for some large $B > 0$. It thus follows by Proposition 1, that

$$\mathcal{E} \ll T(\log T)^B \cdot \sum_{\ell > 0} (a_\ell b_\ell)^{-1/2+\varepsilon}$$

Because of (6) we have $a_\ell b_\ell \gg e^{2\pi\ell}$. Therefore the ℓ's with $\ell \ge (\log\log T)^{1+\varepsilon}$ contribute $\ll_A T(\log T)^{-A}$. We can therefore subsequently assume that $\ell \ll (\log\log T)^{1+\varepsilon}$. In order to control a_ℓ and b_ℓ, when $\ell \le (\log\log T)^{1+\varepsilon}$ we appeal to a result of Waldschmidt (see [11], p. 473),

$$\left| e^{\pi m} - \frac{p}{q} \right| \ge \exp\left(-2^{72} \log(2m) \log p \cdot \log\log p \right). \tag{7}$$

Therefore if condition (6) is satisfied then $e^{2\pi\ell}T^{-1+\varepsilon} \ge \exp(-c(\log\ell) \cdot (\log a_\ell)(\log\log a_\ell))$ Therefore, using that $\ell \le (\log\log T)^{1+\varepsilon}$ we get $(\log a_\ell) \cdot (\log\log a_\ell) \gg \log T/(\log\log T)^\varepsilon$, and hence $\log a_\ell \gg \log T/(\log\log T)^{1+\varepsilon}$. Notice also that (6) implies that $a_\ell b_\ell \gg e^{2\pi\ell}$, so that $\sum_{\ell>0}(a_\ell b_\ell)^{-\alpha} = O_\alpha(1)$ for any $\alpha > 0$. Combining these observations we find

$$\sum_{0<\ell<(\log\log T)^{1+\varepsilon}} (a_\ell b_\ell)^{-1/2+\varepsilon} \ll e^{-c\log T/(\log\log T)^{1+\varepsilon}} \sum_{\ell>0} (a_\ell b_\ell)^{-1/4} \ll e^{-c\log T/(\log\log T)^{1+\varepsilon}}.$$

Thus $\mathcal{E} \ll_A T(\log T)^{-A}$ for any fixed $A > 0$, as desired. □

It is possible to generalize this theorem to other progressions, for example to those for which $2\pi/\alpha$ is algebraic. We refer the reader to [11] for the necessary results in diophantine approximation.

Proof of Theorem 2. Set $B(s) = 1$ in Proposition 1. Then, keeping notation as in Proposition 1, we get

$$\sum_\ell |\zeta(\tfrac{1}{2} + i(\alpha\ell + \beta))|^2 \cdot \phi\left(\frac{\ell}{T}\right) = \int_{\mathbb{R}} |\zeta(\tfrac{1}{2} + i(\alpha t + \beta))|^2 \cdot \phi\left(\frac{t}{T}\right) dt + \mathcal{E}$$

The main term is $\sim \hat{\phi}(0) T \log T$. It remains to understand $\mathcal{E}$.

First case. First suppose that $e^{2\pi\ell/\alpha}$ is irrational for all $\ell > 0$. Since $b(k) = 1$ if

$k = 1$ and $b(k) = 0$ otherwise it is easy to see that $F(a_\ell, b_\ell, t) \ll T \log T$ uniformly in a_ℓ, b_ℓ and $T \le t \le 2T$. Thus,

$$\mathcal{E} \ll T \log T \sum_{\ell > 0} (a_\ell b_\ell)^{-1/2}$$

It remains to show that $\sum_{\ell>0}(a_\ell b_\ell)^{-1/2} = o(1)$ as $T \to \infty$. Let $\varepsilon > 0$ be given. Since $a_\ell b_\ell \gg e^{2\pi \ell/\alpha}$ we can find an A such that $\sum_{\ell > A}(a_\ell b_\ell)^{-1/2} \le \varepsilon$. For the remaining integers $\ell \le A$ notice that $e^{2\pi\ell/\alpha}$ is irrational for each $\ell \le A$. Therefore for each $\ell \le A$,

$$\left| \frac{a_\ell}{b_\ell} - e^{2\pi\ell/\alpha} \right| \le \frac{e^{2\pi\ell/\alpha}}{T^{1-\varepsilon}} \tag{8}$$

implies that $a_\ell b_\ell \to \infty$. It follows that $\sum_{\ell \le A}(a_\ell b_\ell)^{-1/2} \le \varepsilon$ once T is large enough. We conclude that $\sum_{\ell>0}(a_\ell b_\ell)^{-1/2} = o(1)$, and hence that $\mathcal{E} = o(T \log T)$ as desired.

Second case. Now consider the case that $e^{2\pi\ell_0/\alpha}$ is rational for some ℓ_0. Write

$$\alpha = \frac{2\pi \ell_0}{\log(m/n)} \tag{9}$$

with co-prime m and n and $|m|$ minimal. Let k be the maximal positive integer such that $m/n = (r/s)^k$ with r, s co-prime. Then,

$$\alpha = \frac{\ell_0}{k} \cdot \frac{2\pi}{\log(r/s)}.$$

Let $d = (\ell_0, k)$. Note that $d = 1$ since otherwise, we may replace ℓ_0 by ℓ_0/d and m and n by $m^{1/d}$ and $n^{1/d}$ in (9) which contradicts the minimality condition on $|m|$.

For each ℓ divisible by ℓ_0 the integers $a_\ell = m^{\ell/\ell_0}$ and $b_\ell = n^{\ell/\ell_0}$ satisfy (8) because $e^{2\pi\ell/\alpha} = (m/n)^{\ell/\ell_0}$. For the remaining integers ℓ not divisible by ℓ_0, $e^{2\pi\ell/\alpha} = (r/s)^{k\ell/\ell_0}$ is irrational, since $\ell_0 | k\ell$ if and only if $\ell_0 | \ell$. We split $\mathcal{E}$ accordingly

$$\mathcal{E} = 4\mathrm{Re} \sum_{\substack{\ell > 0 \\ \ell_0 | \ell}} H(\ell) + 4\mathrm{Re} \sum_{\substack{\ell > 0 \\ \ell_0 \nmid \ell}} H(\ell)$$

The second sum is $o(T \log T)$ as can be seen by repeating the same argument as in the first case. As for the first sum, we find that for each ℓ divisible by ℓ_0,

$$H(\ell) = 2\mathrm{Re} \left(\frac{(m/n)^{i\beta}}{\sqrt{mn}} \right)^{\ell/\ell_0} \cdot \hat{\phi}(0) T \log T + O\Big(\frac{\ell \log mn}{(mn)^{\ell/2\ell_0}} \cdot T \Big).$$

Therefore

$$\begin{aligned} \sum_{\substack{\ell > 0 \\ \ell_0 | \ell}} H(\ell) &= 2\hat{\phi}(0) T \log T \cdot \sum_{\ell > 0} \left(\frac{(m/n)^{i\beta}}{\sqrt{mn}} \right)^{\ell} + O(T) \\ &= \hat{\phi}(0) T \log T \cdot \frac{2\cos(\beta \log(m/n))\sqrt{mn} - 2}{mn + 1 - 2\sqrt{mn}\cos(\beta \log(m/n))} + O(T) \end{aligned}$$

giving the desired estimate for $\mathcal{E}$. □

3. Proof of Theorem 3

Recall that in the notation of Proposition 1,

$$F(a_\ell, b_\ell, t) := \sum_{r\geqslant 1} \frac{1}{r} \sum_{h,k\leqslant T^\theta} b(k)b(h) \sum_{\substack{m,n\geqslant 1\\ mk=a_\ell r\\ nh=b_\ell r}} W\left(\frac{2\pi mn}{\alpha t+\beta}\right)$$

The lemma below, provides a bound for F when the coefficients $b(n)$ are the coefficients of the mollifiers $M_\theta(s)$, that is

$$b(n) = \mu(n)\cdot\left(1-\frac{\log n}{\log T^\theta}\right)$$

and $b(n) = 0$ for $n > T^\theta$.

Lemma 1. *For any $a_\ell, b_\ell \in \mathbb{N}$ with $(a_\ell, b_\ell) = 1$ and $a_\ell b_\ell > 1$, uniformly in $T \le t \le 2T$, we have that*

$$F(a_\ell, b_\ell, t) \ll (a_\ell b_\ell)^\varepsilon \cdot T(\log T)^{-1+\varepsilon}.$$

Proof. For notational ease, let $N = T^\theta$. We first express the conditions in the sum above in terms of Mellin transforms. To be specific since

$$W(x) = \frac{1}{2\pi}\int_{(\varepsilon)} x^{-w}G(w)\frac{dw}{w}$$

with $G(w)$ rapidly decaying along vertical lines, and such that $G(w) = G(-w)$, $G(0) = 1$, we have

$$S = \frac{1}{2\pi i}\int_{(2)} \sum_{m,n\ge 1}\sum_{h,k\le N} b(h)b(k) \sum_{\substack{r\ge 1\\ nk=b_\ell r\\ mh=a_\ell r}} \frac{1}{r}\left(\frac{\alpha t+\beta}{2\pi mn}\right)^w G(w)\frac{dw}{w}$$

$$= \left(\frac{1}{2\pi i}\right)^3 \int_{(2)}\int_{(2)}\int_{(2)} \sum_{m,n\ge 1}\frac{1}{(mn)^w}\sum_{h,k}\frac{\mu(h)\mu(k)}{h^{z_1}k^{z_2}} \sum_{\substack{r\ge 1\\ nk=b_\ell r\\ mh=a_\ell r}} \frac{1}{r}\left(\frac{\alpha t+\beta}{2\pi}\right)^w G(w)\frac{dw}{w}\frac{N^{z_1}dz_1}{\log N z_1^2}\frac{N^{z_2}dz_2}{\log N z_2^2}.$$

The sum over m, n, h, k and r inside the integral may be factored into an Euler product as

$$\sum_{r\ge 1}\frac{1}{r}\left(\sum_{nk=b_\ell r}\frac{1}{n^w}\frac{\mu(k)}{k^{z_2}}\right)\left(\sum_{mh=a_\ell r}\frac{1}{m^w}\frac{\mu(h)}{h^{z_1}}\right)$$
$$= \prod_p\left(1+\frac{1}{p}\left(\frac{1}{p^w}-\frac{1}{p^{z_2}}\right)\left(\frac{1}{p^w}-\frac{1}{p^{z_1}}\right)\right)F(a_\ell b_\ell, w, z_1, z_2)\eta(w, z_1, z_2).$$

Here $\eta(w, z_1, z_2)$ is an Euler product which is absolutely convergent in the region delimited by $\text{Re } w, \text{Re } z_1, \text{Re } z_2 > -1/2$ and we define

$$F(a_\ell b_\ell, w, z_1, z_2) = \prod_{p^j || a_\ell} \frac{1}{p^{(j-1)w}} \left(\frac{1}{p^w} - \frac{1}{p^{z_1}} \right) \left(1 + \frac{1}{p^{1+w}} \left(\frac{1}{p^w} - \frac{1}{p^{z_2}} \right) \right)$$
$$\prod_{p^j || b_\ell} \frac{1}{p^{(j-1)w}} \left(\frac{1}{p^w} - \frac{1}{p^{z_2}} \right) \left(1 + \frac{1}{p^{1+w}} \left(\frac{1}{p^w} - \frac{1}{p^{z_1}} \right) \right)$$
$$\prod_{p \nmid a_\ell b_\ell} \left(1 + \frac{1}{p} \left(\frac{1}{p^w} - \frac{1}{p^{z_2}} \right) \left(\frac{1}{p^w} - \frac{1}{p^{z_1}} \right) \right)^{-1}.$$

Further, we may write

$$\prod_p \left(1 + \frac{1}{p} \left(\frac{1}{p^w} - \frac{1}{p^{z_2}} \right) \left(\frac{1}{p^w} - \frac{1}{p^{z_1}} \right) \right) \eta(w, z_1, z_2) = \frac{\zeta(1+2w)\zeta(1+z_1+z_2)}{\zeta(1+w+z_1)\zeta(1+w+z_2)} \tilde{\eta}(w, z_1, z_2),$$

where $\tilde{\eta}$ denotes an Euler product which is absolutely convergent in the region delimited by $\text{Re } w, \text{Re } z_1, \text{Re } z_2 > -1/2$ and does not depend on a_ℓ or b_ℓ. Thus,

$$S = \left(\frac{1}{2\pi i} \right)^3 \left(\int_{(2)} \right)^3 \frac{\zeta(1+2w)\zeta(1+z_1+z_2)}{\zeta(1+w+z_!)\zeta(1+w+z_2)} \tilde{\eta}(w, z_1, z_2) F(a_\ell b_\ell, w, z_1, z_2) \left(\frac{\alpha t + \beta}{2\pi} \right)^w$$
$$G(w) \frac{dw}{w} \frac{N^{z_1} dz_1}{\log N z_1^2} \frac{N^{z_2} dz_2}{\log N z_2^2}$$

and shifting contours to $\text{Re } w = -\delta$, $\text{Re } z_1 = \text{Re } z_2 = \delta + \delta^2$ gives, since $\alpha t + \beta \asymp T$,

$$S = I_1 + I_2 + I_3 + O\left(\frac{(a_\ell b_\ell)^\delta N^{2\delta + 2\delta^2}}{T^\delta} \right)$$

with I_1, I_2, I_3 specified below. Since $N < T^{1/2-\varepsilon}$ the error term is $\ll (a_\ell b_\ell)^\varepsilon T^{-\varepsilon}$ provided that δ is chosen small enough. Writing

$$H(z_1, z_2) = \frac{\zeta(1+z_1+z_2)}{\zeta(1+z_1)\zeta(1+z_2)} \tilde{\eta}(0, z_1, z_2) F(a_\ell b_\ell, 0, z_1, z_2)$$

we have

$$I_1 = \frac{\log(\alpha t + \beta)}{2} \frac{1}{(2\pi i)^2} \int_{(1/4)} \int_{(1/4)} H(z_1, z_2) \cdot \frac{N^{z_1} dz_1}{\log N z_1^2} \frac{N^{z_2} dz_2}{\log N z_2^2},$$

$$I_2 = -\frac{1}{2} \frac{1}{(2\pi i)^2} \int_{(1/4)} \int_{(1/4)} \left(\frac{\zeta'}{\zeta}(1+z_1) + \frac{\zeta'}{\zeta}(1+z_2) \right) \cdot H(z_1, z_2) \cdot \frac{N^{z_1} dz_1}{\log N z_1^2} \frac{N^{z_2} dz_2}{\log N z_2^2},$$

and

$$I_3 = \frac{1}{2} \frac{1}{(2\pi i)^2} \int_{(1/4)} \int_{(1/4)} \frac{\left(\frac{d}{dw} \tilde{\eta}(w, z_1, z_2) F(a_\ell b_\ell, w, z_1, z_2) \right)_{w=0}}{\tilde{\eta}(0, z_1, z_2) F(a_\ell b_\ell, 0, z_1, z_2)} \cdot H(z_1, z_2) \cdot$$
$$\cdot \frac{N^{z_1} dz_1}{\log N z_1^2} \frac{N^{z_2} dz_2}{\log N z_2^2}.$$

Bounding the integrals is now a standard exercise. As they can be bounded using the exact same procedure, we will focus our attention to I_1 (note in particular, that I_3 is smaller by a factor of $\log T$ compared with the other integrals).

For ease of notation, write $G(z_1,z_2)=\tilde{\eta}(0,z_1,z_2)F(a_\ell b_\ell,0,z_1,z_2)$. Then

$$I_1=\frac{\log(\alpha t+\beta)}{2}\sum_{n\le N}\frac{1}{n}\frac{1}{(2\pi i)^2}\int_{(1/\log N)}\int_{(1/\log N)}\zeta(1+z_1)^{-1}\zeta(1+z_2)^{-1}G(z_1,z_2)\cdot$$
$$\cdot\left(\frac{N}{n}\right)^{z_1+z_2}\frac{dz_1}{\log N z_1^2}\frac{dz_2}{\log N z_2^2},$$

Let $M=\exp(B(\log\log T)^2)$ for B a parameter to be determined shortly. We split the sum in n above to $n\le N/M$ and $n>N/M$.

If $n>N/M$, then shift both contours to the line with real-part $(\log M)^{-1}$ and bound the integrals trivially. The contribution of terms with $n>N/M$ is

$$\ll \log T(\log M)^5(\log N)^{-2}(a_\ell b_\ell)^\epsilon \ll \frac{(a_\ell b_\ell)^\epsilon}{(\log T)^{1-\epsilon}}.$$

Now, for the terms with $n\le N/M$, first truncate both contours at height $\log^4 T$ with an error $\ll (a_\ell b_\ell)^\epsilon\cdot(\log T)^{-1}$. Since $a_\ell b_\ell>1$, we assume without loss of generality that $a_\ell>1$. This in turn implies that $F(a_\ell b_\ell,0,0,z_2)=0$, so that the integrand is holomorphic at $z_1=0$. From the classical zero free region for $\zeta(s)$, there exists a constant $c>0$ such that $(\zeta(1+z_1))^{-1}<\log(|z_1|+1)$ for Re $z_1\ge -c(\log\log T)^{-1}$ and $|\text{Im } z_1|\le\log^4 T$. We now shift the integral in z_1 to Re $z_1=-c(\log\log T)^{-1}$ with an error $\ll (a_\ell b_\ell)^\varepsilon(\log T)^{-1}$ and bound the remaining integral trivially by

$$M^{\frac{-c}{\log\log T}}\log T\cdot(\log\log T)^2(a_\ell b_\ell)^\epsilon \ll \exp(-cB\log\log T)(\log T)^{1+\epsilon}(a_\ell b_\ell)^\epsilon.$$

The result follows upon picking $B=\frac{2}{c}$. ☐

Proof of Theorem 3. Let $B(s)=M_\theta(s)$ with $0<\theta<\frac{1}{2}$. Inserting the bound in Lemma 1 into Proposition 1 we obtain

$$\mathcal{E}\ll\frac{T}{(\log T)^{1-\varepsilon}}\cdot\sum_{\ell>0}\frac{1}{(a_\ell b_\ell)^{1/2-\varepsilon}}+O(T^{1-\varepsilon}).$$

The sum over $\ell>0$ is rapidly convergent: Because of (6) we have $a_\ell\asymp b_\ell e^{2\pi\ell/\alpha}$ and therefore $a_\ell b_\ell\gg e^{2\pi\ell/\alpha}$. It follows that the sum over $\ell>0$ contributes $O(1)$ and we obtain $\mathcal{E}\ll T(\log T)^{-1+\varepsilon}$ as desired. ☐

4. Proof of Theorem 4

Recall that

$$M_\theta(s):=\sum_{n\ge1}\frac{b(n)}{n^s}$$

with coefficients

$$b(n):=\mu(n)\cdot\left(1-\frac{\log n}{\log T^\theta}\right),$$

for $n\le T^\theta$ and $b(n)=0$ otherwise. Define the mollified first moment as

$$\mathcal{I}:=\sum_\ell\zeta(\tfrac{1}{2}+i(\alpha\ell+\beta))M_\theta(\tfrac{1}{2}+i(\alpha\ell+\beta))\phi\left(\frac{\ell}{T}\right),\qquad(10)$$

and recall that

$$\mathcal{J}:=\sum_\ell|\zeta(\tfrac{1}{2}+i(\alpha\ell+\beta))M_\theta(\tfrac{1}{2}+i(\alpha\ell+\beta))|^2\phi\left(\frac{\ell}{T}\right).$$

By Cauchy-Schwarz and $0 \le \phi \le 1$, we have

$$|\mathcal{I}| \le (P_{\alpha,\beta}(T) \cdot T)^{1/2} \cdot \mathcal{J}^{1/2}.$$

Then our Theorem 4 follows from the following Proposition 3 and Theorem 3.

Proposition 3. *Let $\alpha > 0$, β be real numbers. With $\mathcal{I}$ as defined in (10), and for T large,*

$$|\mathcal{I}| = T\hat{\phi}(0) + O\left(\frac{T}{\log T}\right).$$

Proof. Uniformly in $0 \le t \le 2aT$ we have,

$$\zeta(\tfrac{1}{2}+it) = \sum_{n \le 2aT} \frac{1}{n^{1/2+it}} + O\left(\frac{1}{T^{1/2}}\right),$$

Since in addition $|M(\frac{1}{2}+it)| \ll T^{\theta/2+\epsilon}$ for all t, we get

$$\begin{aligned}
\mathcal{I} &= \sum_{\ell} \sum_{n \le 2\alpha T} \frac{1}{n^{1/2+i(\alpha\ell+\beta)}} \cdot M(\tfrac{1}{2}+i(\alpha\ell+\beta))\phi\left(\frac{\ell}{T}\right) + O\big(T^{\theta/2+1/2+\epsilon}\big) \\
&= \sum_{m \le T^\theta} \frac{b(m)}{\sqrt{m}} \sum_{n \le 2\alpha T} \frac{1}{\sqrt{n}} \cdot (mn)^{-ib} \sum_{\ell} (mn)^{-ia\ell} \phi\left(\frac{\ell}{T}\right) + O(T^{3/4}) \\
&= \sum_{m \le T^\theta} \frac{b(m)}{\sqrt{m}} \sum_{n \le 2\alpha T} \frac{1}{\sqrt{n}} \cdot (mn)^{-ib} \sum_{\ell} T\hat{\phi}\left(T\left(\frac{\alpha\log(mn)}{2\pi} - \ell\right)\right) + O(T^{3/4}),
\end{aligned}$$

by Poisson summation applied to the sum over ℓ.

Note that $\hat{\phi}\,(Tc) \ll_A T^{-A}$ for any $|c| > T^{-1+\epsilon}$,which is an immediate result of $\hat{\phi}$ being a member of the Schwarz class. Hence, the sum above may be restricted to $|\ell| \le \frac{\alpha}{2\pi} \cdot \log(2\alpha T^{1+\theta}) + O(T^{\epsilon-1})$. The terms with $\ell = 0$ contributes a main term of $T\hat{\phi}(0)$ when $mn = 1$, and the terms with other values of mn contributes $O(T^{-A})$.

Now consider $\ell \ne 0$. Terms with $|a\log(mn) - \ell| > T^{\epsilon-1}$ contribute $O(T^{-A})$. Otherwise, suppose that

$$\alpha = \frac{2\pi\ell}{\log n_0} + O(T^{\epsilon-1}) \tag{11}$$

for some integer $n_0 > 1$, and fix such a n_0 The term $mn = n_0$ contributes

$$\frac{T}{n_0^{1/2+ib}} \sum_{m|n_0} b(m)\hat{\phi}\left(T(\alpha\log(n_0) - \ell)\right)$$

for T large. This term is bounded by

$$\ll_a \frac{T}{\log T} \cdot \frac{d(n_0)\log n_0}{\sqrt{n_0}}$$

because $b(m) = \mu(m) + O(\log m/\log T)$ for all m, and thus,

$$\sum_{m|n_0} b(m) \ll \frac{d(n_0)\log n_0}{\log T}$$

For a fixed ℓ, the number of n_0 satifying (11) is bounded by $n_0 T^{-1+\varepsilon} + 1$. Thus the total contribution of all the terms is

$$\ll T\frac{n_0^{1/2+\varepsilon}}{T} \cdot T^{\varepsilon} + T\frac{d(n_0)}{\sqrt{n_0}}\frac{\log n_0}{\log T} \ll T^{3/4+\varepsilon} + T\frac{d(n_0)}{\sqrt{n_0}}\frac{\log n_0}{\log T}.$$

We sum this over all the $|\ell| \ll \log(2\alpha T^{1+\theta})$. Such a short sum does not affect the size of the first term above. As for the second term, since $n_0 \asymp e^{2\pi a\ell}$, the sum over $\ell \neq 0$ is bounded by

$$\frac{T}{\log T} \sum_{|\ell|>0} \frac{|\ell|}{e^{|a\pi\ell|(1-\epsilon)}} \ll \frac{T}{\log T}.$$

From this we have that

$$I = \hat{\phi}(0)T + O\left(\frac{T}{\log T}\right)$$

□

Proof of Theorem 1. Appealing to a result of Balasubramanian, Conrey and Heath-Brown [1] to compute $\int_T^{2T} |(\zeta \cdot M_\theta)(\frac{1}{2}+i(\alpha t+\beta))|^2 dt$, we have by Theorem 3 $\mathcal{J} \leq T \cdot (1+\frac{1}{\theta}+o(1))$. Combining this with the inequality

$$\mathcal{I} \leq (P_{\alpha,\beta}(T) \cdot T)^{1/2} \cdot \mathcal{J}^{1/2}$$

and Proposition 3, we obtain

$$\hat{\phi}(0)T(1+o(1)) \leq (P_{\alpha,\beta}(T) \cdot T)^{1/2} \cdot (T \cdot (\tfrac{1}{\theta}+1+o(1)))^{1/2}$$

Hence,

$$P_{\alpha,\beta}(T) \geq \frac{\theta}{\theta+1}\hat{\phi}(0) + o(1)$$

for all $0 < \theta < \frac{1}{2}$. Now we set $\phi(t) = 1$ for $t \in [1+\epsilon, 2-\epsilon]$ so that $\hat{\phi}(0) \geq 1-2\epsilon$. Letting $\theta \to \frac{1}{2}^-$ and $\epsilon \to 0$, we obtain the claim. □

In order to prove the Corollary we need the lemma below.

Lemma 1. *We have,*

$$\sum_{\ell} |M_\theta(\tfrac{1}{2}+i(\alpha\ell+\beta))|^2 \phi\big(\frac{\ell}{T}\big) \ll T\log T$$

Proof. Using Proposition 2 we find that the above second moment is equal to

$$\int_{\mathbb{R}} |M_\theta(\tfrac{1}{2}+i(\alpha t+\beta))|^2 \phi\big(\frac{t}{T}\big)dt + O\bigg(T\hat{\phi}(0)\sum_{\ell>0}\frac{1}{\sqrt{a_\ell b_\ell}} \cdot |F'(a_\ell, b_\ell)|\bigg)$$

where a_ℓ, b_ℓ denotes for each $\ell > 0$ the unique (if it exists!) couple of co-prime integers such that $a_\ell b_\ell > 1$, $b_\ell < T^{1/2-\varepsilon}e^{-\pi\ell/\alpha}$ and

$$\left|\frac{a_\ell}{b_\ell} - e^{2\pi\ell/\alpha}\right| \leq \frac{e^{2\pi\ell/\alpha}}{T^{1-\varepsilon}}$$

and where

$$F'(a_\ell, b_\ell) = \sum_{r\leq T}\frac{b(a_\ell r)b(b_\ell r)}{r} \ll \log T$$

since the coefficients of M_θ are bounded by 1 in absolute value. Since $\int_{\mathbb{R}} |M_\theta(\frac{1}{2}+i(\alpha t+\beta))|^2\phi(t/T)dt \ll T\log T$ the claim follows. □

Proof of the Corollary. Following [5] let $\mathcal{H}_0$ be the set of integers $T \le \ell \le 2T$ at which,

$$|\zeta(\tfrac{1}{2} + i(\alpha\ell + \beta))| \le \varepsilon(\log \ell)^{-1/2}$$

and let $\mathcal{H}_1$ be the set of integers ℓ at which the reverse inequality holds. Notice that,

$$\mathcal{C}_0 := \Big| \sum_{\ell \in \mathcal{H}_0} \zeta(\tfrac{1}{2} + i(\alpha\ell + \beta)) M_\theta(\tfrac{1}{2} + i(\alpha\ell + \beta)) \phi\big(\frac{\ell}{T}\big) \Big|$$

$$\le \varepsilon(\log T)^{-1/2} T^{1/2} \cdot \Big(\sum_{\ell} |M_\theta(\tfrac{1}{2} + i(\alpha\ell + \beta))|^2 \phi\big(\frac{\ell}{T}\big) \Big)^{1/2} \le C\varepsilon T \hat{\phi}(0)$$

for some absolute constant $C > 0$. Hence by Proposition 3 and the Triangle Inequality,

$$\mathcal{C}_1 := \Big| \sum_{\ell \in \mathcal{H}_1} \zeta(\tfrac{1}{2} + i(\alpha\ell + \beta)) M_\theta(\tfrac{1}{2} + i(\alpha\ell + \beta)) \phi\big(\frac{\ell}{T}\big) \Big| \ge (1 - C\varepsilon)\hat{\phi}(0) T$$

while by Cauchy's inequality,

$$\mathcal{C}_1 \le \Big(\mathrm{Card}(\mathcal{H}_1) \Big)^{1/2} \cdot \Big(\sum_{\ell} |\zeta(\tfrac{1}{2} + i(\alpha\ell + \beta)) M_\theta(\tfrac{1}{2} + i(\alpha\ell + \beta))|^2 \phi\big(\frac{\ell}{T}\big) \Big)^{1/2}$$

As in the proof of Theorem 1, by Theorem 5 and a result of Balasubramanian, Conrey and Heath-Brown, the mollified second moment is $\le T \cdot (1 + 1/\theta + o(1))$ as $T \to \infty$. Thus

$$|\mathcal{H}_1| \ge \hat{\phi}(0) \frac{1 - C\varepsilon}{1 + 1/\theta} T$$

Taking $\theta \to \frac{1}{2}^{-}$ and letting $\phi(t) = 1$ on $t \in [1 + \varepsilon; 2 - \varepsilon]$, so that $\hat{\phi}(0) \ge 1 - 2\varepsilon$ we obtain the claim on taking $\varepsilon \to 0$. □

5. Large and small values: Proof of Theorem 5

Let $0 \le \phi \le 1$ be a smooth function, compactly supported in $[1, 2]$. Let

$$A(s) = \sum_{n \le T} \frac{1}{n^s}$$

and let

$$B(s) = \sum_{n \le N} b(n) n^{-s}$$

be an arbitrary Dirichlet polynomial of length N. Consider,

$$\mathcal{R} := \frac{\sum_{\ell} A(\frac{1}{2} + i(\alpha\ell + \beta)) |B(\frac{1}{2} + i(\alpha\ell + \beta))|^2 \phi(\frac{\ell}{T})}{\sum_{\ell} |B(\frac{1}{2} + i(\alpha\ell + \beta))|^2 \phi(\frac{\ell}{T})}.$$

Following Soundararajan [10], and since $\zeta(\frac{1}{2} + it) = A(\frac{1}{2} + it) + O(t^{-1/2})$,

$$\max_{T \le \ell \le 2T} |\zeta(\tfrac{1}{2} + i(\alpha\ell + \beta))| + O(T^{-1/2}) \ge |\mathcal{R}| \ge \min_{T \le \ell \le 2T} |\zeta(\tfrac{1}{2} + i(\alpha\ell + \beta))| + O(T^{-1/2})$$

Thus, to produce large and small values of ζ at discrete points $\frac{1}{2} + i(a\ell + b)$ it suffices to choose a Dirichlet polynomial B that respectively maximizes/minimizes the ratio $\mathcal{R}$. Fix $\varepsilon > 0$. Consider the set S_1 of tuples (a_ℓ, b_ℓ), with $\ell \le 2 \log T$, such that

$$\Big| \frac{\alpha \log \frac{a_\ell}{b_\ell}}{2\pi} - \ell \Big| \le \frac{1}{T^{1-\varepsilon}}$$

and $a_\ell b_\ell > 1$ and both a_ℓ, b_ℓ are less than $T^{1/2-\varepsilon}$. In particular for each ℓ there is at most one such tuple so $|S_1| \leq 2\log T$. From each tuple in S_1 we pick one prime divisor of a_ℓ and one prime divisor of b_ℓ and put them into a set we call S.

We define our resonator coefficients $r(n)$ by setting $L = \sqrt{\log N \log\log N}$ and

$$r(p) = \frac{L}{\sqrt{p}\log p}$$

when $p \in ([L^2; \exp((\log L)^2)]$ and $p \notin S$. In the remaining cases we let $r(p) = 0$. Note in particular that the resonator coefficients change with T.

We then choose $b(n) = \sqrt{n}r(n)$ or $b(n) = \mu(n)\sqrt{n}r(n)$ depending on whether we want to maximize or minimize the ratio $\mathcal{R}$. For either choice of coefficients we have the following lemma.

Lemma 2. *Write $D(s) = \sum_{n\leq T} \frac{a(n)}{n^s}$ with the coefficients $a(n) \ll 1$. If $N = T^{1/2-\delta}$ with $\delta > 10\varepsilon$, then,*

$$\sum_{\ell} D(\tfrac{1}{2}+i(\alpha\ell+\beta))|B(\tfrac{1}{2}+i(\alpha\ell+\beta))|^2\phi\big(\frac{\ell}{T}\big) = \int_{\mathbb{R}} D(\tfrac{1}{2}+i(\alpha t+\beta))|B(\tfrac{1}{2}+i(\alpha t+\beta))|^2\phi\big(\frac{t}{T}\big)dt$$
$$+ O(T^{1+(1-3\delta)/2+4\epsilon})$$

Proof. By Poisson summation we have,

$$\sum_{\ell} D(\tfrac{1}{2} + i(\alpha\ell + \beta))|B(\tfrac{1}{2} + i(\alpha\ell + \beta)|^2\phi\big(\frac{\ell}{T}\big) =$$
$$= T\sum_{\ell}\sum_{\substack{m,n\leq N\\ h\leq T}} \frac{b(m)b(n)a(h)}{\sqrt{mnh}}\Big(\frac{m}{nh}\Big)^{i\beta}\hat{\phi}\Big(T\Big(\frac{\alpha\log\frac{m}{nh}}{2\pi} - \ell\Big)\Big)$$

The term $\ell = 0$ contributes the main term (the continuous average). It remains to bound the remaining terms $\ell \neq 0$. Since $\hat{\phi}(x) \ll (1 + |x|)^{-A}$ the only surviving terms are those for which,

$$\Big|\frac{\alpha\log\frac{m}{nh}}{2\pi} - \ell\Big| \leq \frac{1}{T^{1-\varepsilon}}$$

which in particular implies that $|\ell| \leq 2\log T$. We split our sum into two ranges, $nh < T^{1/2-\epsilon}$ and $nh > T^{1/2-\epsilon}$.

First range. In the first range, for $(m, nh) = 1$, the real numbers $\log m/(nh)$ are spaced by at least $T^{-1+\varepsilon}$ apart. Among all co-prime tuples with both a_ℓ, b_ℓ less than $T^{1/2-\varepsilon}$ there is at most one tuple satisfying,

$$\Big|\frac{\alpha\log\frac{a_\ell}{b_\ell}}{2\pi} - \ell\Big| \leq \frac{1}{T^{1-\varepsilon}}$$

Grouping the terms m, n, h according to $m = a_\ell r$ and $nh = b_\ell r$, we re-write the first sum sum over the range $nh \leq T^{1/2-\varepsilon}$ as follows,

$$T\sum_{\ell\neq 0}\frac{1}{\sqrt{a_\ell b_\ell}}\sum_{r}\frac{1}{r}\sum_{\substack{m,n\leq N\\ nh\leq T^{1/2-\varepsilon}\\ m=a_\ell r\\ nh=b_\ell r}} b(m)b(n)a(h)\Big(\frac{m}{nh}\Big)^{i\beta}\hat{\phi}\Big(T\Big(\frac{\alpha\log\frac{m}{nh}}{2\pi} - \ell\Big)\Big)$$

However by our choice of r we have $b(a_\ell) = 0$, hence by multiplicativity $b(m) = 0$, and it follows that the above sum is zero.

Second range. We now examine the second range $nh > T^{1/2-\varepsilon}$. The condition $nh > T^{1/2-\varepsilon}$ and $n \le T^{1/2-\delta}$ imply that $h > T^{\delta-\varepsilon}$. For fixed m,n we see that there are at most T^{ε} values of h such that

$$\left|\frac{\alpha \log \frac{m}{nh}}{2\pi} - \ell\right| \le \frac{1}{T^{1-\varepsilon}}$$

Putting this together we have the following bound for the sum over $nh > T^{1/2-\varepsilon}$,

$$T \left| \sum_{\ell \neq 0} \sum_{\substack{m,n \le N, h \le T \\ T^{1/2-\varepsilon} < nh}} \frac{b(m)b(n)a(h)}{\sqrt{mnh}} \left(\frac{m}{nh}\right)^{i\beta} \hat{\phi}\left(T\left(\frac{\alpha \log \frac{m}{nh}}{2\pi} - \ell\right)\right) \right|$$
$$\ll T \sum_{|\ell| \le 2\log T} \sum_{m,n \le N} \frac{|b(m)b(n)|}{\sqrt{mn}} \cdot T^{-\delta/2+\varepsilon} T^{\varepsilon}$$
$$\ll T^{1-\delta/2+3\varepsilon} \cdot N \sum_{n \le N} \frac{|b(m)|^2}{n}$$

Then

$$\sum_{n \le N} \frac{|b(m)|^2}{m} \le \prod_{p \ge L^2} \left(1 + \frac{L^2}{p \log^2 p}\right) \ll T^{\varepsilon}$$

because $L^2 \sum_{p > L^2} p^{-1}(\log p)^{-2} \ll \log N / \log\log N = o(\log T)$. Therefore the sum in the second range is bounded by $T^{1-\delta/2+4\varepsilon} N = T^{1+(1-3\delta)/2+4\varepsilon}$. □

In the above lemma we take $\delta = 1/3 + 4\varepsilon$, so that $N = T^{1/6-4\varepsilon}$ and the error term is negligible (that is $\ll T^{1-\varepsilon}$). Setting consecutively $D(s) = A(s)$ and $D(s) = 1$ we get,

$$\mathcal{R} = \frac{\int_{\mathbb{R}} A(\frac{1}{2} + i(\alpha t + \beta))|B(\frac{1}{2} + i(\alpha t + \beta))|^2 \phi\big(\frac{t}{T}\big) dt}{\int_{\mathbb{R}} |B(\frac{1}{2} + i(\alpha t + \beta))|^2 \phi\big(\frac{t}{T}\big) dt}$$

plus a negligible error term. The above ratio was already worked out by Soundararajan in [10] (see Theorem 2.1). Proceeding in the same way, we obtain that the above ratio is equal to,

$$\mathcal{R} = (1 + o(1)) \prod_{p} \left(1 + \frac{b(p)}{p}\right)$$

Suppose that we were interested in small values, in which case $b(n) = \mu(n)\sqrt{n} r(n)$. Then,

$$\mathcal{R} = (1 + o(1)) \prod_{p \notin S} \left(1 - \frac{L}{p \log p}\right)$$

Since

$$\sum_{p \in S} \frac{L}{p \log p} = \sum_{L^2 \le p \le L^2 + 2\log T} \frac{L}{p \log p} = o\left(\sqrt{\frac{\log N}{\log\log N}}\right)$$

we find that

$$\mathcal{R} = \exp\left(-(1 + o(1))\sqrt{\frac{\log N}{\log\log N}}\right)$$

Recall that $N = T^{1/6-4\varepsilon}$. Letting $\varepsilon \to 0$ we obtain the claim since $\mathcal{R} \ge \min_{T \le \ell \le 2T} |\zeta(\frac{1}{2} + i(\alpha\ell + \beta))| + O(T^{-1/2})$. The large value estimate for the maximum of $\zeta(\frac{1}{2} + i(\alpha\ell + \beta))$ is obtained in exactly the same way by choosing $r(n) = \sqrt{n} r(n)$ instead.

6. Proof of the technical Proposition 1

Let $G(\cdot)$ be an entire function with rapid decay along vertical lines, that is $G(x+iy) \ll |y|^{-A}$ for any fixed x and $A > 0$. Suppose also that $G(-w) = G(w)$, $G(0) = 1$ and $\overline{G(w)} = G(\bar{w})$. An example of such a function is $G(w) = e^{w^2}$. For such a function $G(x)$ we define a smooth function

$$W(x) := \frac{1}{2\pi} \int_{(\varepsilon)} x^{-w} G(w) \cdot \frac{dw}{w}.$$

Notice that W is real.

Lemma 3 (Approximate function equation)**.** *We have, for $T < t < 2T$,*

$$|\zeta(\tfrac{1}{2} + it)|^2 = 2 \sum_{mn < T^{1+\varepsilon}} \frac{1}{\sqrt{mn}} \cdot \left(\frac{m}{n}\right)^{it} W\left(\frac{2\pi mn}{t}\right) + O(T^{-2/3}).$$

Remark. *Of course we could work with the usual smoothing V involving the Gamma factors on the Mellin transform side. We believe the smoothing $W(2\pi mn/t)$ to be (slightly) more transparent.*

Proof. By a standard argument (see [6], Theorem 5.3),

$$|\zeta(\tfrac{1}{2} + it)|^2 = \frac{2}{2\pi i} \int_{(\varepsilon)} \zeta(\tfrac{1}{2} + it + w)\zeta(\tfrac{1}{2} - it + w)\pi^{-w} G(w) \cdot g_t(w) \frac{dw}{w}. \tag{12}$$

with $g_t(w) = \Gamma(\frac{1}{4} + \frac{it}{2} + \frac{w}{2})\Gamma(\frac{1}{4} - \frac{it}{2} + \frac{w}{2}) / \big(\Gamma(\frac{1}{4} + \frac{it}{2})\Gamma(\frac{1}{4} - \frac{it}{2})\big)$ By Stirling's formula $g_t(w) = (t/2)^w \cdot (1 + O((1 + |w|^2)/t))$ uniformly for w lying in any fixed half-plane and t large. Using Weyl's subconvexity bound, on the line $\text{Re } w = \varepsilon$ we have $\zeta(\frac{1}{2} + it + w)\zeta(\frac{1}{2} - it + w) \ll |t|^{1/3} + |w|^{1/3}$. Therefore, the error term $O((1 + |w|^2)/t)$ in Stirling's approximation contributes an error term of $O(T^{-2/3})$ in (12). Thus

$$|\zeta(\tfrac{1}{2} + it)|^2 = \frac{2}{2\pi i} \int_{(\varepsilon)} \zeta(\tfrac{1}{2} + it + w)\zeta(\tfrac{1}{2} - it + w) \cdot \left(\frac{t}{2\pi}\right)^w G(w) \cdot \frac{dw}{w} + O(T^{-2/3}).$$

Shifting the line of integration to $\text{Re } w = 1 + \varepsilon$ we collect a pole at $w = \frac{1}{2} \pm it$, it is negligible because $G(\frac{1}{2} \pm it) \ll |t|^{-A}$. Expanding $\zeta(\frac{1}{2} + it + w)\zeta(\frac{1}{2} - it + w)$ into a Dirichlet series on the line $\text{Re } w = 1 + \varepsilon$ we conclude that

$$|\zeta(\tfrac{1}{2} + it)|^2 = 2 \sum_{m,n \geq 1} \frac{1}{\sqrt{mn}} \cdot \left(\frac{m}{n}\right)^{it} W\left(\frac{2\pi mn}{t}\right) + O(T^{-2/3}).$$

Notice that $W(x) = O_A(x^{-A})$ for $x > 1$. Since $T \leq t \leq 2T$ if $mn > T^{1+\varepsilon}$ then $2\pi mn/t \gg T^{\varepsilon}$. Therefore we can truncate the terms with $mn > T^{1+\varepsilon}$ making an error term of at most $\ll T^{-A}$. The claim follows. □

Recall also that

$$B(s) := \sum_{n \leqslant T^{\theta}} \frac{b(n)}{n^s}$$

Therefore,

$$\begin{aligned}\mathcal{J} &:= \sum_{\ell\in\mathbb{Z}} |\zeta(\tfrac12 + i(\alpha\ell+\beta))B(\tfrac12 + i(\alpha\ell+\beta))|^2 \cdot \phi(\frac{\ell}{T}) \\ &= 2\sum_{mn<T^{1+\varepsilon}} \frac{1}{\sqrt{mn}} \sum_{h,k\leqslant T^\theta} \frac{b(h)b(k)}{\sqrt{hk}} \sum_{\ell\in\mathbb{Z}} \left(\frac{mh}{nk}\right)^{i(\alpha\ell+\beta)} W\left(\frac{2\pi mn}{\alpha\ell+\beta}\right)\phi(\frac{\ell}{T}) + O\big(T^{5/6+\varepsilon}\big)\end{aligned}$$

$$(13) = 2\sum_{mn<T^{1+\varepsilon}} \frac{1}{\sqrt{mn}} \sum_{h,k\leqslant T^\theta} \frac{b(h)b(k)}{\sqrt{hk}} \cdot \left(\frac{mh}{nk}\right)^{i\beta} \sum_{\ell\in\mathbb{Z}} \hat{f}_{m,n,T}\left(\frac{\alpha\log\frac{mh}{nk}}{2\pi} - \ell\right) + O\big(T^{5/6+\varepsilon}\big)$$

using Poisson summation in the sum over ℓ, with

$$f_{m,n,T}(x) := W\left(\frac{2\pi mn}{\alpha x+\beta}\right)\cdot\phi(\frac{x}{T})$$

6.1. **The main term $\ell = 0$.** Consider the sum with $\ell = 0$,

$$\begin{aligned}&2\sum_{mn<T^{1+\varepsilon}} \frac{1}{\sqrt{mn}} \sum_{h,k\leqslant T^\theta} \frac{b(h)b(k)}{\sqrt{hk}} \cdot \left(\frac{mk}{nh}\right)^{i\beta} \cdot \hat{f}_{m,n,T}\left(\frac{\alpha\log\frac{mk}{nh}}{2\pi}\right) \\ &= 2\sum_{mn<T^{1+\varepsilon}} \frac{1}{\sqrt{mn}} \sum_{h,k\leq T^\theta} \frac{b(h)b(k)}{\sqrt{hk}} \cdot \int_{\mathbb{R}} \left(\frac{mk}{nh}\right)^{i(\alpha t+\beta)} W\left(\frac{2\pi mn}{\alpha t+\beta}\right)\phi\left(\frac{t}{T}\right)dt\end{aligned}$$

Interchanging the sums and the integral, this becomes

$$(14) \qquad \int_{\mathbb{R}} |B(\tfrac12 + i(\alpha t+\beta))|^2 \cdot 2\sum_{mn<T^{1+\varepsilon}} \frac{1}{\sqrt{mn}} \cdot \left(\frac{m}{n}\right)^{i(\alpha t+\beta)} W\left(\frac{2\pi mn}{\alpha t+\beta}\right)\phi(\frac{t}{T})dt$$

By the approximate functional equation,

$$2\sum_{mn<T^{1+\varepsilon}} \frac{1}{\sqrt{mn}} \left(\frac{m}{n}\right)^{i(\alpha t+\beta)} W\left(\frac{2\pi mn}{\alpha t+\beta}\right) = |\zeta(\tfrac12 + i(\alpha t+\beta)|^2 + O(T^{-2/3}).$$

Therefore (14) is

$$\int_{\mathbb{R}} |B(\tfrac12 + i(\alpha t+\beta))\zeta(\tfrac12 + i(\alpha t+\beta))|^2 \phi\left(\frac{t}{T}\right)dt + O(T^{1-\varepsilon})$$

as desired.

6.2. **The terms $\ell \neq 0$.** Since

$$\hat{f}_{m,n,T}\left(\frac{\alpha\log\frac{mh}{nk}}{2\pi} + \ell\right) = \overline{\hat{f}_{m,n,T}\left(\frac{\alpha\log\frac{nk}{mh}}{2\pi} - \ell\right)}$$

we can re-write the sum over $\ell \neq 0$ so as to have $\ell > 0$ in the summation,

$$\mathcal{J}_0 = 2\sum_{\ell>0} \sum_{\substack{mn<T^{1+\varepsilon}\\ h,k\leq T^\theta}} \frac{b(h)b(k)}{\sqrt{mnhk}} \cdot 2\mathrm{Re}\left(\left(\frac{mh}{nk}\right)^{i\beta} \hat{f}_{m,n,T}\left(\frac{\alpha\log\frac{mh}{nk}}{2\pi} - \ell\right)\right)$$

Differentiating repeatedly and using that W and all derivatives of W are Schwarz class, we find that for $mn < T^{1+\varepsilon}$, $f^{(k)}_{m,n,T}(x) \ll T^{-k}$ for all x. Therefore for any fixed $A > 0$,

$$\hat{f}_{m,n,T}(x) \ll_A T\big(1 + T|x|\big)^{-A}$$

It follows that the only integers m, n, k, h, ℓ that contribute to $\mathcal{J}_0$ are the m, n, k, h, ℓ for which

$$\left| \frac{\alpha}{2\pi} \cdot \log \frac{mh}{nk} - \ell \right| \leq T^{-1+\eta}.$$

for some small, but arbitrary $\eta > 0$. This condition implies that

$$\left| \frac{mh}{nk} - e^{2\pi\ell/\alpha} \right| \leq e^{2\pi\ell/\alpha} T^{-1+\eta} \tag{15}$$

and we might as-well restrict the sum in $\mathcal{J}_0$ to those m, n, k, h, ℓ satisfying this weaker, but friendlier, condition. Thus,

$$\mathcal{J}_0 = 4\mathrm{Re} \sum_{\ell > 0} \sum_{\substack{mn < T^{1+\varepsilon} \\ h,k \leq T^{\theta} \\ m,n,h,k \text{ satisfy (15)}}} \frac{b(h)b(k)}{\sqrt{mnhk}} \cdot \left(\frac{mk}{nh} \right)^{i\beta} \hat{f}_{m,n,T} \left(\frac{\alpha \log \frac{mk}{nh}}{2\pi} - \ell \right) + O_A \left(\frac{1}{T^A} \right). \tag{16}$$

Now for a fixed $\ell > 0$, consider the inner sum over m, n, h, k in (16). We group together terms in the following way: If the integres m, n, k, h satisfy (15) then we let $a_\ell = mk/(mk, nh)$ and $b_\ell = nh/(mk, nh)$ so that $(a_\ell, b_\ell) = 1$. We group together all multiples of a_ℓ, b_ℓ of the form $mk = a_\ell r$ and $nh = b_\ell r$ with a common $r > 0$. The a_ℓ, b_ℓ are co-prime and satisfy

$$\left| \frac{a_\ell}{b_\ell} - e^{2\pi\ell/\alpha} \right| < \frac{e^{2\pi\ell/\alpha}}{T^{1-\eta}}. \tag{17}$$

This allows us to write

$$\mathcal{J}_0 = 4\mathrm{Re} \sum_{\ell > 0} \sum_{\substack{a_\ell, b_\ell \geqslant 1 \\ (a_\ell, b_\ell) = 1 \\ \text{satisfy (17)}}} \sum_{r \geqslant 1} \sum_{\substack{mn \leqslant T^{1+\varepsilon} \\ h,k \leqslant T^{\theta} \\ nh = a_\ell r \\ mk = b_\ell r}} \frac{b(h)b(k)}{\sqrt{mknh}} \cdot \left(\frac{a_\ell}{b_\ell} \right)^{\mathrm{i}\beta} \cdot \hat{f}_{m,n,T} \left(\frac{\alpha \log(a_\ell / b_\ell)}{2\pi} - \ell \right). \tag{18}$$

It is useful to have a bound for the size of b_ℓ in the above sum. Equation (17) implies that $a_\ell \asymp b_\ell \cdot e^{2\pi\ell/\alpha}$. Furthermore, since $mn < T^{1+\varepsilon}$, $h, k \leqslant T^{\theta}$ and $a_\ell r = mk$, $b_\ell r = nh$ we have $a_\ell \cdot b_\ell < mnkh < T^{1+2\theta+\varepsilon}$. Combining $a_\ell \asymp b_\ell \cdot e^{2\pi\ell/\alpha}$ and $a_\ell b_\ell < T^{1+2\theta+\varepsilon}$ we obtain $b_\ell < T^{1/2+\theta+\varepsilon} \cdot e^{-\pi\ell/\alpha}$. Let

$$K_\ell := T^{1/2-\eta} e^{-\pi\ell/\alpha}$$
$$M_\ell := T^{1/2+\theta+\varepsilon} e^{-\pi\ell/\alpha}$$

We split the sum according to whether $b_\ell < K_\ell$ or $b_\ell > K_\ell$, getting

$$\mathcal{J}_0 = 4\mathrm{Re} \sum_{\ell > 0} \sum_{\substack{b_\ell < M_\ell \\ a_\ell \geqslant 1 \\ (a_\ell, b_\ell) = 1 \\ \text{satisfy (17)}}} \frac{(a_\ell / b_\ell)^{\mathrm{i}\beta}}{\sqrt{a_\ell b_\ell}} \sum_{r \geqslant 1} \frac{1}{r} \sum_{\substack{mn \leqslant T^{1+\varepsilon} \\ h,k \leqslant T^{\theta} \\ nh = b_\ell r \\ mk = a_\ell r}} \hat{f}_{m,n,T} \left(\frac{\alpha \log(a_\ell / b_\ell)}{2\pi} - \ell \right) = 4\mathrm{Re}\ (S_1 + S_2)$$

where S_1 is the sum over $b_\ell \leq K_\ell$ and S_2 is the corresponding sum over $M_\ell > b_\ell > K_\ell$. To finish the proof of the Proposition it remains to evaluate S_1 and S_2. The sum S_1 can give a main term contribution in the context of Theorem 2 depending on the Diophantine properties of a, while bounding S_1 as an error term in the context of Theorem 4 is relatively subtle. In contrast, S_2 is always negligible.

We first furnish the following expression for S_1.

Lemma 4. *For each $\ell > 0$ there is at most one tuple of co-prime integers (a_ℓ, b_ℓ) such that $a_\ell b_\ell > 1$, $b_\ell < K_\ell = T^{1/2-\eta}e^{-\pi\ell/\alpha}$ and such that*

$$\left|\frac{a_\ell}{b_\ell} - e^{2\pi\ell/\alpha}\right| \le \frac{e^{2\pi\ell/\alpha}}{T^{1-\eta}}. \tag{19}$$

We denote by $\sum_\ell^*$ the sum over ℓ's satisfying the above condition. Then,

$$S_1 = T \cdot \sum_{\ell>0}{}^{*} \frac{(a_\ell/b_\ell)^{i\beta}}{\sqrt{a_\ell b_\ell}} \int_{-\infty}^{\infty} \phi\Big(\frac{t}{T}\Big) \cdot \exp\bigg(-2\pi i t\Big(\frac{\alpha \log \frac{a_\ell}{b_\ell}}{2\pi} - \ell\Big)\bigg) \cdot F(a_\ell, b_\ell, t) dt$$

where

$$\begin{aligned} F(a_\ell, b_\ell, t) &:= \sum_{h,k\le T^\theta} b(h)b(k) \sum_{r\ge 1} \frac{1}{r} \sum_{\substack{m,n\ge 1 \\ mk=a_\ell r \\ nh=b_\ell r}} W\Big(\frac{\alpha t+\beta}{2\pi mn}\Big) \\ &= \sum_{m,n\le T^\theta} \frac{b(m)b(n)}{mn} \cdot (ma_\ell, nb_\ell) \cdot \mathcal{H}\Big((\alpha t+\beta)\cdot \frac{(ma_\ell, nb_\ell)^2}{2\pi m a_\ell n b_\ell}\Big) \end{aligned}$$

and

$$\mathcal{H}(x) = \frac{1}{2\pi i}\int_{(\varepsilon)} \zeta(1+2w)\cdot x^w G(w) \cdot \frac{dw}{w} = \begin{cases} \frac{1}{2}\cdot \log x + \gamma + O_A(x^{-A}) & \text{if } x \gg 1 \\ O_A(x^A) & \text{if } x \ll 1 \end{cases}$$

Proof Given ℓ, there is at most one $b_\ell \leqslant K_\ell$ for which there is a co-prime a_ℓ such that (19) holds, because Farey fractions with denominator $< K_\ell$ are spaced at least $K_\ell^{-2} = e^{2\pi\ell/\alpha}T^{-1+2\eta}$ far apart. Thus for each ℓ, the sum over a_ℓ, b_ℓ in S_1 consists of at most one element (a_ℓ, b_ℓ),

$$S_1 = \sum_{\ell>0}{}^{*} \frac{(a_\ell/b_\ell)^{ib}}{\sqrt{a_\ell b_\ell}} \sum_{r\geqslant 1}\frac{1}{r} \sum_{\substack{mn\leqslant T^{1+\varepsilon} \\ h,k\leqslant T^\theta \\ nh=b_\ell r \\ mk=a_\ell r}} \frac{b(h)b(k)}{\sqrt{mknh}} \cdot \hat{f}_{m,n,T}\Big(\frac{\alpha\log(a_\ell/b_\ell)}{2\pi} - \ell\Big)$$

To simplify the above expression we write

$$\hat{f}_{m,n,T}(x) \;=\; \int_{-\infty}^{\infty} W\Big(\frac{2\pi mn}{\alpha t+\beta}\Big)\phi\Big(\frac{t}{T}\Big)e^{-2\pi \mathrm{i} xt}\mathrm{d}t$$

The sum S_1 can be now re-written as,

$$T\sum_{\ell>0}{}^{*} \frac{(a_\ell/b_\ell)^{ib}}{\sqrt{a_\ell b_\ell}} \int_{-\infty}^{\infty}\phi\Big(\frac{t}{T}\Big)\exp\bigg(-2\pi i t\Big(\frac{\alpha\log\frac{mh}{nk}}{2\pi} - \ell\Big)\bigg)\cdot$$
$$\sum_{r\geqslant 1}\frac{1}{r}\sum_{h,k\leqslant T^\theta} b(h)b(k) \sum_{\substack{mn<T^{1+\varepsilon} \\ nh=b_\ell r, mk=a_\ell r}} W\Big(\frac{2\pi mn}{\alpha t+\beta}\Big)\mathrm{d}t.$$

Since $W(x) \ll x^{-A}$ for $x > 1$ and $at + b \asymp T$ we complete the sum over $mn < T^{1+\varepsilon}$ to $m, n \geqslant 1$ making a negligible error term $\ll_A T^{-A}$. To finish the proof it remains to understand the expression

$$\sum_{h,k\le T^\theta} b(h)b(k)\sum_r \frac{1}{r}\sum_{\substack{m,n\ge 1 \\ mk=a_\ell r \\ nh=b_\ell r}} W\Big(\frac{2\pi mn}{\alpha t+\beta}\Big) \tag{20}$$

We notice that

$$\sum_r \frac{1}{r} \sum_{\substack{m,n\geq 1\\ nh=b_\ell r\\ mk=a_\ell r}} W\left(\frac{2\pi mn}{\alpha t+\beta}\right) = \frac{1}{2\pi}\int_{(\varepsilon)} \sum_r \frac{1}{r} \sum_{\substack{m,n\geq 1\\ nh=b_\ell r\\ mk=a_\ell r}} \frac{1}{(mn)^w}\cdot\left(\frac{\alpha t+\beta}{2\pi}\right)^w G(w)\frac{dw}{w} \tag{21}$$

Furthermore $nh = b_\ell r$ and $mk = a_\ell r$ imply that $mkb_\ell = nha_\ell$. On the other hand since a_ℓ and b_ℓ are co-prime the equality $mkb_\ell = nha_\ell$ implies that there exists a unique r such that $nh = b_\ell r$ and $mk = a_\ell r$. We notice as-well that this unique r can be expressed as $((a_\ell b_\ell)/(mknh))^{-1/2}$. Therefore we have the equality,

$$\sum_r \frac{1}{r} \sum_{\substack{m,n\geq 1\\ nh=b_\ell r\\ mk=a_\ell r}} \frac{1}{(mn)^w} = \sum_{\substack{m,n\geq 1\\ nha_\ell=mkb_\ell}} \frac{1}{(mn)^w}\cdot\sqrt{\frac{a_\ell b_\ell}{mknh}}$$

We express the condition $nha_\ell = mkb_\ell$ as $ha_\ell | kb_\ell m$ and $n = kb_\ell m/(ha_\ell)$ so as to reduce the double sum over m, n to a single sum over m. Furthermore the condition $ha_\ell | kb_\ell m$ can be dealt with by noticing that it is equivalent to $ha_\ell/(ha_\ell, kb_\ell) | m$. Using these observations we find that,

$$\sum_{\substack{m,n\geq 1\\ nha_\ell=mkb_\ell}} \frac{1}{(mn)^w}\cdot\sqrt{\frac{a_\ell b_\ell}{mknh}} = \frac{(ha_\ell, kb_\ell)}{hk}\cdot\zeta(1+2w)\cdot\left(\frac{(ha_\ell, kb_\ell)^2}{ha_\ell kb_\ell}\right)^w$$

Plugging the above equation into (21) it follows that

$$\sum_{r\geq 1}\frac{1}{r}\sum_{\substack{m,n\geq 1\\ mk=b_\ell r\\ nh=a_\ell r}} W\left(\frac{2\pi mn}{\alpha t+\beta}\right) = \frac{(ha_\ell, kb_\ell)}{hk}\cdot\mathcal{H}\left(\frac{\alpha t+\beta}{2\pi mn}\right)$$

An easy calculation reveals that $\mathcal{H}(x) = (1/2)\log x + \gamma + O_A(x^{-A})$ for $x \gg 1$ and that $\mathcal{H}(x) = O_A(x^A)$ for $x \ll 1$. We conclude that equation (20) equals to

$$\sum_{h,k\leq T^\theta} \frac{b(h)b(k)}{hk}\cdot(ha_\ell, kb_\ell)\cdot\mathcal{H}\left(\frac{\alpha t+\beta}{2\pi mn}\right)$$

as desired. □

The second sum S_2 can be bounded directly.

Lemma 5. *We have* $S_2 \ll T^{1/2+\theta+\varepsilon}$.

Proof Recall that the a_ℓ, b_ℓ are always assumed to satisfy the condition

$$\left|\frac{a_\ell}{b_\ell} - e^{2\pi\ell/\alpha}\right| \leq \frac{e^{2\pi\ell/\alpha}}{T^{1-\eta}}. \tag{22}$$

Recall also that

$$K_\ell := T^{1/2-\eta}e^{-\pi\ell/\alpha}$$
$$M_\ell := T^{1/2+\theta+\varepsilon}e^{-\pi\ell/\alpha}$$

Then,

$$S_2 = \sum_{\ell\in\mathbb{Z}} \sum_{\substack{K_\ell<b_\ell<M_\ell\\ a_\ell\geqslant 1\\ (a_\ell,b_\ell)=1\\ \text{satisfy (22)}}} \frac{(a_\ell/b_\ell)^{i\beta}}{\sqrt{a_\ell b_\ell}} \sum_{r\geqslant 1}\frac{1}{r}\sum_{h,k\leqslant T^\theta} b(h)b(k) \sum_{\substack{mn\leqslant T^{1+\varepsilon}\\ nh=b_\ell r\\ mk=a_\ell r}} \hat{f}_{m,n,T}\left(\frac{\alpha\log(a_\ell/b_\ell)}{2\pi} - \ell\right). \tag{23}$$

We split the above sum into dyadic blocks $b_\ell \asymp N$ with $K_\ell < N < M_\ell$. The number of $(a_\ell, b_\ell) = 1$ with $b_\ell \asymp N$ and satisfying (22) is bounded by

$$\ll \frac{e^{2\pi\ell/\alpha}}{T^{1-\eta}} \cdot N^2 + 1$$

because Farey fractions with denominators of size $\asymp N$ are spaced at least N^{-2} apart. Therefore, for a fixed ℓ, using the bounds $b(n) \ll n^\varepsilon$ and $\hat{f}_{m,n,T}(x) \ll T$, the dyadic block with $b_\ell \asymp N$ contributes at most,

$$\ll T^{1+\varepsilon} \sum_{\substack{b_\ell \asymp N \\ a_\ell \geqslant 1 \\ (a_\ell, b_\ell)=1 \\ a_\ell, b_\ell \text{satisfy (22)}}} \frac{1}{(a_\ell b_\ell)^{1/2}} \sum_{r<T^2} \frac{1}{r} \sum_{\substack{m,n,h,k \\ mk=a_\ell r \\ nh=b_\ell r}} 1 \ll T^{1+\varepsilon} \sum_{\substack{b_\ell \asymp N \\ a_\ell \geqslant 1 \\ (a_\ell, b_\ell)=1 \\ a_\ell, b_\ell \text{ satisfy (22)}}} \frac{1}{(a_\ell b_\ell)^{1/2-\varepsilon}} \tag{24}$$

because

$$\sum_{r \leq T^2} \frac{1}{r} \sum_{\substack{m,h,k,n \\ mh=a_\ell r \\ nk=b_\ell r}} 1 = \sum_{r \leq T^2} \frac{d(a_\ell r) d(b_\ell r)}{r} \ll (T a_\ell b_\ell)^\varepsilon.$$

Since $a_\ell \asymp b_\ell \cdot e^{2\pi\ell/\alpha}$ the sum (24) is bounded by,

$$\ll \frac{T}{N} \cdot (TN)^\varepsilon e^{-(1-\varepsilon)\pi\ell/\alpha} \cdot \left(\frac{e^{2\pi\ell/\alpha}}{T^{1-\eta}} \cdot N^2 + 1 \right).$$

Keeping ℓ fixed and summing over all possible dyadic blocks $K_\ell < N < M_\ell$ shows that for fixed ℓ the inner sum in (23) is bounded by

$$\begin{aligned} &\ll T^{\varepsilon+\eta} \cdot e^{\pi(1+\varepsilon)\ell/\alpha} \cdot M_\ell^{1+\varepsilon} + T^{1+\varepsilon} \cdot K_\ell^{-1+\varepsilon} \cdot e^{-(1-\varepsilon)\pi\ell/\alpha} \\ &\ll T^{1/2+\theta+\varepsilon+\eta} \cdot e^{\varepsilon\ell/\alpha} + T^{1/2+\eta+\varepsilon} \cdot e^{\varepsilon\ell/\alpha}. \end{aligned} \tag{25}$$

The condition (22) restricts ℓ to $0 < \ell < 2\alpha \log T$. Summing (25) over all $0 < \ell < 2\alpha \log T$ we find that S_2 is bounded by $T^{1/2+\theta+2\varepsilon+\eta} + T^{1/2+\eta+\varepsilon}$. Since $\theta < \frac{1}{2}$ and we can take η, ε arbitrarily small, but fixed, the claim follows. □

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