SELBERG'S CENTRAL LIMIT THEOREM FOR $\log|\zeta(\frac{1}{2}+it)|$

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1. Introduction

In this paper we give a new and simple proof of Selberg's influential theorem [8, 9] that $\log|\zeta(\frac{1}{2}+it)|$ has an approximately normal distribution with mean zero and variance $\frac{1}{2}\log\log|t|$. Apart from some basic facts about the Riemann zeta function, we have tried to make our proof self-contained.

Theorem 1. *Let V be a fixed real number. Then for all large T,*

$$\frac{1}{T}meas\Big\{T \le t \le 2T : \quad \log|\zeta(\tfrac{1}{2}+it)| \ge V\sqrt{\tfrac{1}{2}\log\log T}\Big\} \sim \frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-u^2/2}du.$$

We outline the steps of the proof. The first step is to show that $\log|\zeta(\frac{1}{2}+it)|$ is usually close to $\log|\zeta(\sigma+it)|$ for suitable σ near $\frac{1}{2}$.

Proposition 1. *Let T be large, and suppose $T \le t \le 2T$. Then for any $\sigma > 1/2$ we have*

$$\int_{t-1}^{t+1}\Big|\log|\zeta(\tfrac{1}{2}+iy)| - \log|\zeta(\sigma+iy)|\Big|dy \ll (\sigma-\tfrac{1}{2})\log T.$$

The proof of Proposition 1 is the only place where we will briefly need to mention the zeros of $\zeta(s)$. From now on, we set $\sigma_0 = \frac{1}{2}+\frac{W}{\log T}$, for a suitable parameter $W \ge 3$ to be chosen later. From Proposition 1, and in view of the Theorem 1 that we set out to prove, we see that if $W = o(\sqrt{\log\log T})$ then we may typically approximate $\log|\zeta(\frac{1}{2}+it)|$ by $\log|\zeta(\sigma_0+it)|$. Thus we may from now on focus on the distribution of $\log|\zeta(\sigma_0+it)|$.

There is considerable latitude in choosing parameters such as W, but to fix ideas we select

$$W = (\log\log\log T)^4, \quad X = T^{1/(\log\log\log T)^2}, \text{ and } Y = T^{1/(\log\log T)^2}. \tag{1}$$

Here X and Y are two parameters that will appear shortly. Put

$$\mathcal{P}(s) = \mathcal{P}(s;X) = \sum_{2\le n\le X}\frac{\Lambda(n)}{n^s\log n}. \tag{2}$$

By computing moments, it is not hard to determine the distribution of $\mathcal{P}(s)$.

Proposition 2. *As t varies in $T \le t \le 2T$, the distribution of $Re(\mathcal{P}(\sigma_0+it))$ is approximately normal with mean 0 and variance $\sim \frac{1}{2}\log\log T$.*

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Our goal is now to connect $\mathrm{Re}(\mathcal{P}(\sigma_0 + it))$ with $\log|\zeta(\sigma_0 + it)|$ for most values of t. This is done in two stages. First we introduce a Dirichlet polynomial $M(s)$ which we show is typically close to $\exp(-\mathcal{P}(s))$. Define $a(n) = 1$ if n is composed of only primes below X, and it has at most $100 \log\log T$ primes below Y, and at most $100 \log\log\log T$ primes between Y and X; set $a(n) = 0$ in all other cases. Put

$$M(s) = \sum_n \frac{\mu(n)a(n)}{n^s}. \tag{3}$$

Note that $a(n) = 0$ unless $n \le Y^{100 \log\log T} X^{100 \log\log\log T} < T^{\epsilon}$, and so $M(s)$ is a short Dirichlet polynomial.

Proposition 3. *With notations as above, we have for $T \le t \le 2T$*

$$M(\sigma_0 + it) = (1 + o(1)) \exp(-\mathcal{P}(\sigma_0 + it)),$$

except perhaps on a subset of measure $o(T)$.

The final step of the proof shows that $\zeta(\sigma_0 + it)M(\sigma_0 + it)$ is typically close to 1.

Proposition 4. *With notations as above,*

$$\int_T^{2T} |1 - \zeta(\sigma_0 + it)M(\sigma_0 + it)|^2 dt = o(1),$$

so that for $T \le t \le 2T$ we have

$$\zeta(\sigma_0 + it)M(\sigma_0 + it) = 1 + o(1),$$

except perhaps on a set of measure $o(T)$.

Proof of Theorem 1. To recapitulate the argument, Proposition 4 shows that typically $\zeta(\sigma_0 + it) \approx M(\sigma_0 + it)^{-1}$, which by Proposition 3 is $\approx \exp(\mathcal{P}(\sigma_0 + it))$, and therefore by Proposition 2 we may conclude that $\log|\zeta(\sigma_0 + it)|$ is normally distributed. Finally by Proposition 1 we deduce from this the normal distribution of $\log|\zeta(\frac{1}{2} + it)|$. This completes the proof of Theorem 1. $\square$

After developing the proofs of the propositions, in Section 7 we compare and contrast our approach with previous proofs, and also discuss possible extensions of this technique.

2. Proof of Proposition 1

Put $G(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)$ and $\xi(s) = G(s)\zeta(s)$ denote the completed ζ-function. If t is large and $t - 1 \le y \le t + 1$, then by Stirling's formula $|\log G(\sigma + iy)/G(1/2 + iy)| \ll (\sigma - 1/2)\log t$, and so it is enough to prove that

$$\int_{t-1}^{t+1} \Big| \log \Big| \frac{\xi(\frac{1}{2} + iy)}{\xi(\sigma + iy)} \Big| \Big| dy \ll (\sigma - \tfrac{1}{2}) \log T.$$

Recall Hadamard's factorization formula

$$\xi(s) = e^{A+Bs}\prod_\rho \Big(1-\frac{s}{\rho}\Big)e^{s/\rho},$$

where A and B are constants with $B=-\sum_\rho \mathrm{Re}\,(1/\rho)$. Thus (assuming that y is not the ordinate of a zero of $\zeta(s)$)

$$\log\Big|\frac{\xi(\frac12+iy)}{\xi(\sigma+iy)}\Big| = \sum_\rho \log\Big|\frac{\frac12+iy-\rho}{\sigma+iy-\rho}\Big|.$$

Integrating the above over $y\in(t-1,t+1)$ we get

$$\int_{t-1}^{t+1}\Big|\log\Big|\frac{\xi(\frac12+iy)}{\xi(\sigma+iy)}\Big|\Big|dy \le \sum_\rho \int_{t-1}^{t+1}\Big|\log\Big|\frac{\frac12+iy-\rho}{\sigma+iy-\rho}\Big|\Big|dy. \tag{4}$$

Suppose $\rho=\beta+i\gamma$ is a zero of $\zeta(s)$. If $|t-\gamma|\ge 2$ then we check readily that

$$\int_{t-1}^{t+1}\Big|\log\Big|\frac{\frac12+iy-\rho}{\sigma+iy-\rho}\Big|\Big|dy \ll \frac{(\sigma-\frac12)}{(t-\gamma)^2}.$$

In the range $|t-\gamma|\le 2$ we use

$$\int_{t-1}^{t+1}\Big|\log\Big|\frac{\frac12+iy-\rho}{\sigma+iy-\rho}\Big|\Big|dy \le \frac12\int_{-\infty}^{\infty}\Big|\log\frac{(\beta-\frac12)^2+x^2}{(\beta-\sigma)^2+x^2}\Big|dx = \pi(\sigma-\tfrac12).$$

Thus in either case

$$\int_{t-1}^{t+1}\Big|\log\Big|\frac{\frac12+iy-\rho}{\sigma+iy-\rho}\Big|\Big|dy \ll \frac{(\sigma-\frac12)}{1+(t-\gamma)^2}.$$

Inserting this in (4), and noting that there are $\ll \log(t+k)$ zeros with $k\le|t-\gamma|<k+1$, the proposition follows.

3. Proof of Proposition 2

We begin by showing that we may restrict the sum in $\mathcal{P}(s)$ just to primes. The contribution of cubes and higher powers of primes is clearly $O(1)$, and we need only discard the contribution of squares of primes. By integrating out, it is easy to see that

$$\int_T^{2T}\Big|\sum_{p\le\sqrt{X}}\frac{1}{2p^{2(\sigma_0+it)}}\Big|^2 dt \ll \sum_{p_1,p_2\le\sqrt{X}}\min\Big(T,\frac{1}{|\log(p_1/p_2)|}\Big)\ll T.$$

Therefore, the measure of the set $t\in[T,2T]$ with the contribution of prime squares being larger than L (say) is at most $\ll T/L^2$. In view of this, to establish Proposition 2, it is enough to prove that

$$\mathcal{P}_0(\sigma_0+it) := \mathrm{Re}\sum_{p\le X}\frac{1}{p^{\sigma_0+it}}$$

has an approximately Gaussian distribution with mean 0 and variance $\sim \frac{1}{2}\log\log T$. We establish this by computing moments.

Lemma 1. *Suppose that k and ℓ are non-negative integers with $X^{k+\ell} \le T$. Then, if $k \neq \ell$,*

$$\int_T^{2T} \mathcal{P}_0(\sigma_0+it)^k \mathcal{P}_0(\sigma_0-it)^\ell dt \ll T,$$

while if $k=\ell$ we have

$$\int_T^{2T} |\mathcal{P}_0(\sigma_0+it)|^{2k} dt = k! T(\log\log T)^k + O_k(T(\log\log T)^{k-1+\epsilon}).$$

Proof. Write $\mathcal{P}(s)^k = \sum_n a_k(n) n^{-s}$, where $a_k(n)=0$ unless n has the prime factorization $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ where $p_1, \ldots, p_r$ are distinct primes below X, and $\alpha_1+\ldots+\alpha_r = k$, in which case $a_k(n) = k!/(\alpha_1!\cdots\alpha_r!)$. Therefore, expanding out the integral, we obtain

$$\int_T^{2T} \mathcal{P}_0(\sigma_0+it)^k \mathcal{P}_0(\sigma_0-it)^\ell dt = T\sum_n \frac{a_k(n)a_\ell(n)}{n^{2\sigma_0}} + O\Big(\sum_{m\neq n} \frac{a_k(m)a_\ell(n)}{(mn)^{\sigma_0}} \frac{1}{|\log(m/n)|}\Big).$$

If $m \neq n$, then $|\log(m/n)| \ge 1/\sqrt{mn}$ and so the off-diagonal terms above contribute $\ll \sum_{m\neq n} a_k(m)a_\ell(n) \ll X^{k+\ell}$. Note that if $k \neq \ell$ then $a_k(n)a_\ell(n)$ is always zero, and the first statement of the lemma follows.

It remains in the case $k=\ell$ to discuss the diagonal term $\sum_n a_k(n)^2/n^{2\sigma_0}$. The terms with n not being square-free are easily seen to contribute $O_k((\log\log T)^{k-2})$. Finally the square-free terms n give

$$k! \sum_{\substack{p_1,\ldots,p_k \le X \\ p_j \text{ distinct}}} \frac{1}{(p_1\cdots p_k)^{2\sigma_0}} = k!\Big(\sum_{p\le X} \frac{1}{p^{2\sigma_0}}\Big)^k + O_k((\log\log T)^{k-1}),$$

and the lemma follows. □

From Lemma 1 we see that if $X^k \le T$ then for odd k

$$\int_T^{2T} (\text{Re } \mathcal{P}_0(\sigma_0+it)^k dt \ll T,$$

while if k is even then

$$\frac{1}{T}\int_T^{2T} (\text{Re } \mathcal{P}_0(\sigma_0+it)^k dt = 2^{-k}\binom{k}{k/2}(k/2)!(\log\log T)^{k/2} + O_k((\log\log T)^{k-1+\epsilon}).$$

These moments match the moments of a Gaussian random variable with mean zero and variance $\sim \frac{1}{2}\log\log T$, and since the Gaussian is determined by its moments, our proposition follows.

4. Proof of Proposition 3

Let us decompose $\mathcal{P}(s)$ as $\mathcal{P}_1(s)+\mathcal{P}_2(s)$, where

$$\mathcal{P}_1(s) = \sum_{2\le n\le Y} \frac{\Lambda(n)}{n^s \log n}, \qquad \text{and} \qquad \mathcal{P}_2(s) = \sum_{Y<n\le X} \frac{\Lambda(n)}{n^s\log n}.$$

Put

$$\mathcal{M}_1(s) = \sum_{0\le k\le 100\log\log T} \frac{(-1)^k}{k!}\mathcal{P}_1(s)^k, \qquad \text{and} \qquad \mathcal{M}_2(s) = \sum_{0\le k\le 100\log\log\log T} \frac{(-1)^k}{k!}\mathcal{P}_2(s)^k.$$

Lemma 2. *For $T\le t\le 2T$ we have*

$$|\mathcal{P}_1(\sigma_0+it)|\le \log\log T, \ \ \textit{and}\ |\mathcal{P}_2(\sigma_0+it)|\le \log\log\log T, \tag{5}$$

except perhaps for a set of measure $\ll T/\log\log\log T$. When the bounds (5) hold, we have

$$\mathcal{M}_1(\sigma_0+it) = \exp(-\mathcal{P}_1(\sigma_0+it))\Big(1+O((\log T)^{-99})\Big), \tag{6}$$

and

$$\mathcal{M}_2(\sigma_0+it) = \exp(-\mathcal{P}_2(\sigma_0+it))\Big(1+O((\log\log T)^{-99})\Big). \tag{7}$$

Proof. Note that

$$\int_T^{2T} |\mathcal{P}_1(\sigma_0+it)|^2 dt \ll \sum_{2\le n_1,n_2\le Y} \frac{\Lambda(n_1)\Lambda(n_2)}{(n_1n_2)^{\sigma_0}\log n_1\log n_2} \min\Big(T, \frac{1}{|\log(n_1/n_2)|}\Big) \ll T\log\log T,$$

and similarly

$$\int_T^{2T} |\mathcal{P}_2(\sigma_0+it)|^2 dt \ll T\log\log\log T.$$

The first assertion (5) follows.

If $|z|\le K$ then using Stirling's formula it is straightforward to check that

$$\Big| e^z - \sum_{0\le k\le 100K} \frac{z^k}{k!}\Big| \le e^{-99K},$$

and therefore the estimates (6) and (7) hold. $\square$

Put $a_1(n)=1$ if n is composed of at most $100\log\log T$ primes all below Y, and zero otherwise. Put $a_2(n)=1$ if n is composed of at most $100\log\log\log T$ primes all between Y and X, and zero otherwise. Then $M(s)=M_1(s)M_2(s)$ with

$$M_1(s) = \sum_n \frac{\mu(n)a_1(n)}{n^s} \qquad \text{and} \qquad M_2(s) = \sum_n \frac{\mu(n)a_2(n)}{n^s}.$$

Lemma 3. *With notations as above, we have*

$$\int_T^{2T} |\mathcal{M}_1(\sigma_0+it) - M_1(\sigma_0+it)|^2 dt \ll T(\log T)^{-60},$$

and

$$\int_T^{2T} |\mathcal{M}_2(\sigma_0+it) - M_2(\sigma_0+it)|^2 dt \ll T(\log\log T)^{-60}.$$

Proof. We establish the first estimate, and the second follows similarly. If we expand $\mathcal{M}_1(s)$ into a Dirichlet series $\sum_n b(n)n^{-s}$, then we may see that $|b(n)| \le 1$ for all n, $b(n) = 0$ unless $n \le Y^{100\log\log T}$ is composed only of primes below Y, and $b(n) = \mu(n)a_1(n)$ if $\Omega(n) \le 100\log\log T$. (It is the presence of prime powers in $\mathcal{P}(s)$ that prevents $\mathcal{M}_1(s)$ from simply being $M_1(s)$.) Thus, putting $c(n) = b(n) - \mu(n)a_1(n)$ temporarily, we see that

$$\int_T^{2T} |\mathcal{M}_1(\sigma_0+it) - M_1(\sigma_0+it)|^2 dt \ll \sum_{n_1,n_2} \frac{|c(n_1)c(n_2)|}{(n_1n_2)^{\sigma_0}} \min\Big(T, \frac{1}{|\log(n_1/n_2)|}\Big).$$

The terms with $n_1 \ne n_2$ contribute (since $|\log(n_1/n_2)| \gg 1/\sqrt{n_1n_2}$ in that case)

$$\ll \sum_{n_1\ne n_2 \le Y^{100\log\log T}} 1 \ll T^{\epsilon}.$$

The diagonal terms $n_1 = n_2$ contribute, for any $1 < r < 2$,

$$\ll T \sum_{\substack{p|n \implies p\le Y \\ \Omega(n) > 100\log\log T}} \frac{1}{n} \ll T r^{-100\log\log T} \prod_{p\le Y} \Big(1 + \frac{r}{p} + \frac{r^2}{p^2} + \dots\Big).$$

Choosing $r = e^{2/3}$, say, the above is $\ll T(\log T)^{-60}$. □

Proof of Proposition 3. From Lemma 3 it follows that except on a set of measure $o(T)$, one has $M_1(\sigma_0+it) = \mathcal{M}_1(\sigma_0+it) + O((\log T)^{-25})$. Moreover, from (6) (except on a set of measure $o(T)$) we note that $\mathcal{M}_1(\sigma_0+it) = \exp(-\mathcal{P}_1(\sigma_0+it))(1+O((\log T)^{-99}))$, and by (5) that $(\log T)^{-1} \ll |\mathcal{M}_1(\sigma_0+it)| \ll \log T$. Therefore, we may conclude that, except on a set of measure $o(T)$,

$$M_1(\sigma_0+it) = \mathcal{M}_1(\sigma_0+it) + O((\log T)^{-25}) = \exp(-\mathcal{P}_1(\sigma_0+it))(1+O((\log T)^{-20})).$$

Similarly, except on a set of measure $o(T)$, we have

$$M_2(\sigma_0+it) = \mathcal{M}_2(\sigma_0+it) + O((\log\log T)^{-25}) = \exp(-\mathcal{P}_2(\sigma_0+it))(1+O((\log\log T)^{-20})).$$

Multiplying these estimates we obtain

$$M(\sigma_0+it) = \exp(-\mathcal{P}(\sigma_0+it))(1+O((\log\log T)^{-20})),$$

completing our proof. □

5. Proof of Proposition 4

For $T \le t \le 2T$, one has $\zeta(\sigma_0+it) = \sum_{n\le T} n^{-\sigma_0-it} + O(T^{-\frac{1}{2}})$, and so

$$\int_T^{2T} \zeta(\sigma_0+it)M(\sigma_0+it)dt = \sum_{n\le T}\sum_m \frac{a(m)\mu(m)}{(mn)^\sigma}\int_T^{2T}(mn)^{-it}dt + O(T^{\frac{1}{2}+\epsilon}) = T + O(T^{\frac{1}{2}+\epsilon}).$$

Therefore, expanding the square, we see that

$$\int_T^{2T} |1-\zeta(\sigma_0+it)M(\sigma_0+it)|^2 dt = \int_T^{2T} |\zeta(\sigma_0+it)M(\sigma_0+it)|^2dt - T + O(T^{\frac{1}{2}+\epsilon}). \tag{8}$$

It remains to evaluate the integral above, and to do this we shall use the following familiar lemma (see for example Lemma 6 of Selberg [8]). For completeness we include a quick proof of the lemma in the next section, and we note that we give only a version sufficient for our purposes and not the sharpest known result.

Lemma 4. *Let h and k be non-negative integers, with $h,k\le T$. Then, for any $1\ge\sigma>\frac{1}{2}$,*

$$\int_T^{2T}\Big(\frac{h}{k}\Big)^{it}|\zeta(\sigma+it)|^2dt = \int_T^{2T}\Big(\zeta(2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{\sigma} + \Big(\frac{t}{2\pi}\Big)^{1-2\sigma}\zeta(2-2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{1-\sigma}\Big)dt$$
$$+ O(T^{1-\sigma+\epsilon}\min(h,k)).$$

Assuming Lemma 4, we now complete the proof of Proposition 4. In view of (8) we must show that

$$\sum_{h,k}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{\sigma_0}}\int_T^{2T}\Big(\frac{h}{k}\Big)^{it}|\zeta(\sigma_0+it)|^2 \sim T, \tag{9}$$

and to do this we appeal to Lemma 4. The error terms are easily seen to be $o(T)$, and we now focus on the main terms arising from Lemma 4, beginning with the first main term. This contributes

$$T\zeta(2\sigma_0)\sum_{h,k}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{2\sigma_0}}(h,k)^{2\sigma_0}. \tag{10}$$

Write $h=h_1h_2$ where h_1 is composed only of primes below Y, and h_2 of primes between Y and X, and then $a(h)=a_1(h_1)a_2(h_2)$ in the notation of section 3. Writing similarly $a(k)=a_1(k_1)a_2(k_2)$, we see that the quantity in (10) factors as

$$T\zeta(2\sigma_0)\Big(\sum_{h_1,k_1}\frac{\mu(h_1)\mu(k_1)a_1(h_1)a_1(k_1)}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0}\Big)\Big(\sum_{h_2,k_2}\frac{\mu(h_2)\mu(k_2)a_2(h_2)a_2(k_2)}{(h_2k_2)^{2\sigma_0}}(h_2,k_2)^{2\sigma_0}\Big). \tag{11}$$

Consider the first factor in (11). If we ignore the condition that h_1 and k_1 must have at most $100 \log\log T$ prime factors, then the resulting sum is simply

$$\sum_{\substack{h_1,k_1 \\ p|h_1k_1 \implies p\le Y}} \frac{\mu(h_1)\mu(k_1)}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0} = \prod_{p\le Y}\Big(1-\frac{1}{p^{2\sigma_0}}\Big).$$

In approximating the first factor by the product above, we incur an error term which is at most (by symmetry we may suppose that h_1 has many prime factors)

$$\begin{aligned}
&\ll \sum_{\substack{h_1,k_1 \\ p|h_1k_1 \implies p\le Y \\ \Omega(h_1)>100\log\log T}} \frac{|\mu(h_1)\mu(k_1)|}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0} \\
&\ll e^{-100\log\log T} \sum_{\substack{h_1,k_1 \\ p|h_1k_1 \implies p\le Y}} \frac{|\mu(h_1)\mu(k_1)|}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0} e^{\Omega(h_1)} \\
&\ll (\log T)^{-100}\prod_{p\le Y}\Big(1+\frac{1+2e}{p}\Big) \ll (\log T)^{-90}.
\end{aligned}$$

Similarly one obtains that the second factor in (11) is

$$\prod_{Y<p\le X}\Big(1-\frac{1}{p^{2\sigma_0}}\Big) + O((\log\log T)^{-90}).$$

Using these in (11), we obtain that the first main term is

$$\sim T\zeta(2\sigma_0)\prod_{p\le X}\Big(1-\frac{1}{p^{2\sigma_0}}\Big) = T\prod_{p>X}\Big(1-\frac{1}{p^{2\sigma_0}}\Big)^{-1} \sim T$$

since $(\sigma_0 - \frac{1}{2})^{-1} = o(\log T/\log X)$. In the same way we see that the second main term arising from Lemma 4 is

$$\begin{aligned}
&\zeta(2-2\sigma_0)\Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^{1-2\sigma_0}dt\Big)\sum_{h,k}\frac{\mu(h)\mu(k)a(h)a(k)}{hk}(h,k)^{2-2\sigma_0} \\
&\sim \Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^{1-2\sigma_0}dt\Big)\zeta(2-2\sigma_0)\prod_{p\le X}\Big(1-\frac{2}{p}+\frac{1}{p^{2\sigma_0}}\Big) = o(T).
\end{aligned}$$

This completes our proof of (9), and hence of Proposition 4.

6. Proof of Lemma 4

Put $G(s) = \pi^{-s/2}s(s-1)\Gamma(s/2)$, so that $\xi(s) = G(s)\zeta(s) = \xi(1-s)$ is the completed zeta function. Define for any given $s \in \mathbb{C}$

$$I(s) = I(\overline{s}) = \frac{1}{2\pi i}\int_{(c)} \xi(z+s)\xi(z+\overline{s})e^{z^2}\frac{dz}{z},$$

where the integral is over the line from $c-i\infty$ to $c+i\infty$ for any $c>0$. By moving the line of integration to the left, and using the functional equation $\xi(z+s)\xi(z+\overline{s}) = \xi(-z+(1-s))\xi(-z+(1-\overline{s}))$ we obtain that

$$|\zeta(s)|^2 = \frac{1}{|G(s)|^2}\Big(I(s)+I(1-s)\Big). \tag{12}$$

From now on suppose that $s=\sigma+it$ with $T\le t\le 2T$, and $1\ge\sigma\ge\frac12$. If z is a complex number with real part $c=1-\sigma+1/\log T$, then an application of Stirling's formula gives

$$\frac{G(z+s)G(z+\overline{s})}{|G(s)|^2} = \Big(\frac{t}{2\pi}\Big)^z\Big(1+O\Big(\frac{|z|}{T}\Big)\Big).$$

Therefore, we see that

$$\frac{I(s)}{|G(s)|^2} = \frac{1}{2\pi i}\int_{(1-\sigma+1/\log T)}\Big(\frac{t}{2\pi}\Big)^z\zeta(z+s)\zeta(z+\overline{s})e^{z^2}\frac{dz}{z}+O(T^{-\sigma+\epsilon}).$$

Since we are in the region of absolute convergence of $\zeta(z+s)$ and $\zeta(z+\overline{s})$, we obtain

$$\begin{aligned}\int_T^{2T}\Big(\frac{h}{k}\Big)^{it}\frac{I(s)}{|G(s)|^2}dt = \frac{1}{2\pi i}\int_{(1-\sigma+1/\log T)}\frac{e^{z^2}}{z}\sum_{m,n=1}^{\infty}\frac{1}{(mn)^{z+\sigma}}\Big(\int_T^{2T}\Big(\frac{hm}{kn}\Big)^{it}\Big(\frac{t}{2\pi}\Big)^z dt\Big)dz\\ +O(T^{1-\sigma+\epsilon}).\end{aligned} \tag{13}$$

In the integral in (13), we distinguish the diagonal terms $hm=kn$ from the off-diagonal terms $hm\ne kn$. The diagonal terms $hm=kn$ may be parametrized as $m=Nk/(h,k)$ and $n=Nh/(h,k)$, and therefore these terms contribute

$$\frac{1}{2\pi i}\int_{(1-\sigma+1/\log T)}\frac{e^{z^2}}{z}\zeta(2z+2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{z+\sigma}\Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^z dt\Big)dz. \tag{14}$$

As for the off-diagonal terms, the inner integral over t may be bounded by $\ll T^{1-\sigma}\min(T,1/|\log(hm/kn)|)$, and therefore these contribute

$$\ll T^{1-\sigma}\sum_{\substack{m,n=1\\ hm\ne kn}}^{\infty}\frac{1}{(mn)^{1+1/\log T}}\min\Big(T,\frac{1}{|\log(hm/kn)|}\Big)\ll\min(h,k)T^{1-\sigma+\epsilon}. \tag{15}$$

The final estimate above follows by first discarding terms with $hm/(kn)>2$ or $<1/2$, and for the remaining terms (assume that $k\le h$) noting that for a given m the sum over values n may be bounded by kT^ϵ (here it may be useful to distinguish the cases $hm>T$ and $hm<T$).

From (13), (14) and (15), we conclude that

$$\begin{aligned}\int_T^{2T}\Big(\frac{h}{k}\Big)^{it}\frac{I(s)}{|G(s)|^2}dt = \frac{1}{2\pi i}\int_{(1-\sigma+1/\log T)}\frac{e^{z^2}}{z}\zeta(2z+2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{z+\sigma}\Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^z dt\Big)dz\\ +O(\min(h,k)T^{1-\sigma+\epsilon}).\end{aligned} \tag{16}$$

A similar argument gives

$$\int_T^{2T} \Big(\frac{h}{k}\Big)^{it} \frac{I(1-s)}{|G(s)|^2} dt = O(T^{1-\sigma+\epsilon}\min(h,k))$$
$$+ \frac{1}{2\pi i}\int_{(\sigma+1/\log T)} \frac{e^{z^2}}{z}\zeta(2z+2-2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{z+1-\sigma}\Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^{z+1-2\sigma}dt\Big)dz. \tag{17}$$

With a suitable change of variables, we can combine the main terms in (16) and (17) as

$$\frac{1}{2\pi i}\int_{(1+1/\log T)} \zeta(2z)\Big(\frac{(h,k)^2}{hk}\Big)^{z}\Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^{z-\sigma}dt\Big)\Big(\frac{e^{(z-\sigma)^2}}{z-\sigma}+\frac{e^{(z-1+\sigma)^2}}{z-1+\sigma}\Big)dz,$$

and moving the line of integration to the left we obtain the main term of the lemma as the residues of the poles at $z=\sigma$ and $z=1-\sigma$ (note that the potential pole at $z=1/2$ from $\zeta(2z)$ is canceled by a zero of $e^{(z-\sigma)^2}/(z-\sigma)+e^{(z-1+\sigma)^2}/(z-1+\sigma)$ there). This completes our proof of Lemma 4.

7. Discussion

In common with Selberg's proof of Theorem 1 our proof relies on the Gaussian distribution of short sums over primes, as in Proposition 2. In contrast with Selberg's proof, we do not need to invoke zero density estimates for $\zeta(s)$, the easier mean-value theorem in Proposition 4 provides for us a sufficient substitute. Selberg's original proof also used an intricate argument expressing $\log\zeta(s)$ in terms of primes and zeros; an elegant alternative approach was given by Bombieri and Hejhal [1], although they too require a strong zero density result near the critical line. We should also point out that by just focussing on the central limit theorem, we have not obtained asymptotic formulae for the moments of $\log|\zeta(\frac12+it)|$ which Selberg established.

In Selberg's approach, it was easier to handle $\operatorname{Im}\log\zeta(\frac12+it)$, and the case of $\log|\zeta(\frac12+it)|$ entailed additional technicalities. In contrast, our method works well for $\log|\zeta(\frac12+it)|$ but requires substantial modifications to handle $\operatorname{Im}\log\zeta(\frac12+it)$. The reason is that Proposition 4 guarantees that typically $|\zeta(\sigma_0+it)|\approx|M(\sigma_0+it)|^{-1}$, but it could be that $\operatorname{Im}\log\zeta(\frac12+it)$ and $\operatorname{Im}\log M(\sigma_0+it)^{-1}$ are not typically close but differ by a substantial integer multiple of 2π. In this respect our argument shares some similarities with Laurinčikas's proof of Selberg's central limit theorem [6], which relies on bounding small moments of $|\zeta(\frac12+it)|$ using Heath-Brown's work on fractional moments [4]. In particular, Laurinčikas's argument also breaks down for the imaginary part of $\log\zeta(\frac12+it)$.

We can quantify the argument given here, providing a rate of convergence to the limiting distribution. With more effort (in particular taking higher moments in Lemma 2 to obtain better bounds on the exceptional set there) we can recover previous results in this direction, but have not been able to obtain anything stronger. We also remark that the argument also gives the joint distribution of $\log|\zeta(\frac12+it)|$ and $\log|\zeta(\frac12+it+i\alpha)|$ (for any fixed non-zero $\alpha\in\mathbb{R}$) and shows that these are distributed like independent Gaussians. One can allow for more than one shift, and also keep track of the uniformity in α.

Our proof of Proposition 3 (in Section 4) involved splitting the mollifier $M(s)$ into two factors, or equivalently of the prime sum $\mathcal{P}(s)$ into two pieces. We would have liked to get away with just one prime sum, but this barely fails. In order to use Proposition 1 successfully, we are forced to take $W = o(\sqrt{\log\log T})$. To mollify successfully on the $\frac{1}{2} + \frac{W}{\log T}$ line (see Proposition 4) we need to work with primes going up to roughly $T^{\frac{1}{W}}$. If $W = o(\sqrt{\log\log T})$ then this length is $T^{A/\sqrt{\log\log T}}$ for a large parameter A, and if we try to expand $\exp(\mathcal{P}(s))$ into a series (as in Section 4) we will be forced to take more than $\sqrt{\log\log T}$ terms in the exponential series. This leads to Dirichlet polynomials that are just a little too long. We resolve this (see Section 4) by splitting $\mathcal{P}$ into two terms, exploiting the fact that the longer sum $\mathcal{P}_2$ has a significantly smaller variance.

Propositions 1, 2, and 3 in our argument are quite general and analogs may be established for higher degree L-functions in the t-aspect. An analog of Proposition 4 however can at present only be established for L-functions of degree 2 (relying here upon information on the shifted convolution problem), and unknown for degrees 3 or higher. However, some hybrid results are possible. For example, by adapting the techniques in [2, 3] we can establish an analog of Proposition 4 for twists of a fixed $GL(3)$ L-function by primitive Dirichlet characters with conductor below Q. In this way one can show that as χ ranges over all primitive Dirichlet characters with conductor below Q, and t ranges between -1 and 1, the distribution of $\log|L(\frac{1}{2}+it, f\times\chi)|$ is approximately normal with mean 0 and variance $\sim \frac{1}{2}\log\log Q$; here f is a fixed eigenform on $GL(3)$.

Keating and Snaith [5] have conjectured that central values of L-functions in families have a log normal distribution with an appropriate mean and variance depending on the family. For example, we may consider the family of quadratic Dirichlet L-functions $L(\frac{1}{2}, \chi_d)$ where d ranges over fundamental discriminants of size X. In this setting, we may carry out the arguments of Propositions 2, 3 and 4 and conclude that $\log L(\sigma_0, \chi_d)$ has a normal distribution with mean $\sim \frac{1}{2}\log\log X$ and variance $\sim \log\log X$, provided that $\sigma_0 = \frac{1}{2} + \frac{W}{\log X}$ where W is any function with $W \to \infty$ as $X \to \infty$ and with $\log W = o(\log\log X)$. However in this situation we do not have an analog of Proposition 1 allowing us to pass from this to the central value; indeed, our knowledge at present does not exclude the possibility that $L(\frac{1}{2}, \chi_d) = 0$ for a positive proportion of discriminants d.

Finally we remark that the proof presented here was suggested by earlier work of the authors [7], where general one sided central limit theorems towards the Keating-Snaith conjectures are established.

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