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ERRATUM TO "COCOMPACTLY CUBULATED GRAPH MANIFOLDS" ISRAEL JOURNAL OF MATHEMATICS **207** (2015), 377–394

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The following lemma is Lemma 4.7 of [HP15]. In the proof of part (2), we incorrectly invoked [CS11, Prop. 2.6]. Here we correct the proof, emphasising that the statement is unchanged. The current proof is largely a re-writing of the proof of [HJP15, Lem 6.2].

Lemma: *Consider the product of the free cyclic and a finitely generated non-abelian free group $H = \mathbb{Z} \times \mathbb{F}$. Suppose that H acts freely and cocompactly on a* CAT(0) *cube complex $\mathcal{V}$. Then the following hold:*

(1) *The essential core $\mathcal{V}^{ess}$ of $\mathcal{V}$ is a product $\mathcal{V}_a \times \mathcal{V}_b$, where $\mathcal{V}_a, \mathcal{V}_b$ are unbounded.*
(2) *The group H has a finite-index subgroup $H' = H_a \times H_b$ that preserves the above decomposition, where H_a acts trivially on $\mathcal{V}_b$ and H_b acts trivially on $\mathcal{V}_a$.*
(3) *We have $H_a = \mathbb{Z} \cap H'$ and the group H_b embeds as a finite-index subgroup of the free group $H/\mathbb{Z} \cong \mathbb{F}$ under the natural quotient.*

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In the proof we will use the fact that any fixed-point-free isometry g of a finite-dimensional cube complex $\mathcal{V}$ is **hyperbolic**, which means that $\inf_{v\in\mathcal{V}} d(gv,v)$ is attained and non-zero [Bri99a, Thm A]. We denote by $\mathrm{Min}(g)\subset\mathcal{V}$ the set on which the infimum is attained. If $\mathcal{V}$ is CAT(0), then by [Bri99b, Thm II.6.8] we have a product decomposition $\mathrm{Min}(g)=\mathbb{R}\times Y$, where each $\mathbb{R}\times\{y\}$ is a g-invariant geodesic line called an **axis** of g. Moreover, each isometry h commuting with g preserves $\mathrm{Min}(g)$ and respects its product decomposition.

Proof. Since H is a direct product with infinite factors, no element is rank-one in the action on $\mathcal{V}^{ess}$. Corollary 6.4(iii) of [CS11] yields a nontrivial cubical product decomposition of $\mathcal{V}^{ess}$ such that each factor has some $h\in H$ respecting the decomposition and acting on that factor as a rank-one isometry. This proves (1). By [CS11, Prop 2.6], there is a finite index subgroup $H'\leq H$ respecting this decomposition. Let $\mathcal{V}_a$ be a factor on which $H_a=H'\cap\mathbb{Z}$ acts freely. Combine all other factors into $\mathcal{V}_b$, so $\mathcal{V}^{ess}=\mathcal{V}_a\times\mathcal{V}_b$.

We claim that the generator z of H_a acts on $\mathcal{V}_a$ as a rank-one isometry. Otherwise, let $h\in H$ be the element guaranteed by [CS11, Cor 6.4(iii)] respecting the decomposition and acting on $\mathcal{V}_a$ as a rank-one isometry. Then the axes of h are not parallel to the axes of z. Hence $\langle h,z\rangle\cong\mathbb{Z}^2$, and this subgroup acts properly on $\mathcal{V}_a$, contradicting the fact that h is rank-one. This justifies the claim.

Consider $\mathrm{Min}(z)=\mathbb{R}\times Y\subset\mathcal{V}_a$. Since z is central in H', we have an induced action of H' on $\mathbb{R}\times Y$ respecting this decomposition. Since z is rank-one, Y does not contain a geodesic ray, and hence is bounded. Consequently, Y contains a fixed point of the action of H', whence $\mathcal{V}_a$ contains an H'-invariant line l.

Let $\rho: H'\to\mathrm{Isom}(l)$ be the induced map and note that $\rho(H')$ does not switch the ends of l. Since $\mathcal{V}_a$ is a cube complex, the translation lengths on l are discrete. Thus $\rho(H')$ can be identified with the integers, containing $\rho(H_a)$ as a finite index subgroup. Let $H_b=\ker(\rho)$. Since $H_b\cap H_a=\{1\}$, the subgroup H_b embeds into $H/\mathbb{Z}=\mathbb{F}$; its image has finite index since $H'\leq H$ has finite index. Thus H_b is free, of rank ≥ 2. Replace H' with its finite-index subgroup $H_a\times H_b$.

Choose $a\in l$. Since H_b fixes a, it acts properly on $\{a\}\times\mathcal{V}_b\subset\mathcal{V}^{ess}$, so it acts properly on $\mathcal{V}_b$. By hypothesis, there is a compact set $K\subset\mathcal{V}^{ess}$ such that $H'K=\mathcal{V}^{ess}$. Since H_a acts properly on $\mathcal{V}_a$, there are finitely many $h_a\in H_a$ for which there exists $h_b\in H_b$ so that h_ah_bK intersects $\{a\}\times\mathcal{V}_b$. Hence H_b

acts cocompactly on $\{a\} \times \mathcal{V}_b$, and thus on $\mathcal{V}_b$, so any orbit map $H_b \to \mathcal{V}_b$ is a quasi-isometry.

Let $h, h' \in H_b$ be non-commuting. The sets $\mathrm{Min}(h), \mathrm{Min}(h') \subset \mathcal{V}_b$ have the form $\mathbb{R} \times U, \mathbb{R} \times U'$, where U, U' are bounded (since $\mathcal{V}_b$ is quasi-isometric to a tree). Let R be large enough so that the neighbourhoods $\mathcal{N}_R(\mathrm{Min}(h))$ and $\mathcal{N}_R(\mathrm{Min}(h'))$ intersect. Since z commutes with h and h', it preserves both $\mathrm{Min}(h)$ and $\mathrm{Min}(h')$. Thus H_a preserves $\mathcal{N}_R(\mathrm{Min}(h)) \cap \mathcal{N}_R(\mathrm{Min}(h'))$, which is bounded. Hence H_a fixes some $b \in \mathcal{V}_b$, and consequently the entire orbit $H_b b$ is fixed by H_a. Thus H_a moves each $v \in \mathcal{V}_b$ a uniformly bounded distance.

Above, we obtained an H_a-invariant fiber $\mathcal{V}_a \times \{b\} \subset \mathcal{V}^{ess}$. As before, because $H_a \times H_b$ acts cocompactly on $\mathcal{V}^{ess}$, and H_b acts properly on $\mathcal{V}_b$, we have that H_a acts cocompactly on $\mathcal{V}_a$, so $l \hookrightarrow \mathcal{V}_a$ is a quasi-isometry. Since H_b fixes l, we have that H_b moves each point of $\mathcal{V}_a$ a uniformly bounded distance. Since $\mathcal{V}_a$ and $\mathcal{V}_b$ are locally finite, there is a uniform bound n on the size of each orbit of the action of H_b on $\mathcal{V}_a$ and the action of H_a on $\mathcal{V}_b$. We replace H_a and H_b (and hence H') by the intersection of all their subgroups of index $\leq n!$. Then H_a acts trivially on $\mathcal{V}_b$ and H_b acts trivially on $\mathcal{V}_a$. This proves (2). Along the way we also established (3). ∎

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References

[Bri99a] M. R. Bridson, *On the semisimplicity of polyhedral isometries*, Proceedings of the American Mathematical Society **127** (1999), 2143–2146.

[Bri99b] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften, Vol. 319, Springer-Verlag, Berlin, 1999.

[CS11] P.-E. Caprace and M. Sageev, *Rank rigidity for CAT(0) cube complexes*, Geometric and Functional Analysis **21** (2011), 851–891.

[HJP15] J. Huang, K. Jankiewicz and P. Przytycki, *Cocompactly cubulated 2-dimensional Artin groups*, arXiv e-print 1510.08493, (2015).

[HP15] M. F. Hagen and P. Przytycki, *Cocompactly cubulated graph manifolds*, Israel Journal of Mathematics **207** (2015), 377–394.