A STEP TOWARDS TWIST CONJECTURE

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Abstract. Under the assumption that a defining graph of a Coxeter group admits only twists in $\mathbb{Z}_2$ and is of type FC, we prove Mühlherr's Twist Conjecture.

1. Introduction

A *Coxeter generating set* S of a group W is a set such that (W, S) is a Coxeter system. This means that S generates W subject only to relations of the form $s^2 = 1$ for $s \in S$ and $(st)^{m_{st}} = 1$, where $m_{st} = m_{ts} \geq 2$ for $s \neq t \in S$ (possibly there is no relation between s and t, and then we put by convention $m_{st} = \infty$). An S-*reflection* (or a *reflection*, if the dependence on S does not need to be emphasised) is an element of W conjugate to some element of S. We say that S is *reflection-compatible* with another Coxeter generating set S' if every S-reflection is an S'-reflection. Furthermore, S is *angle-compatible* with S' if for every $s, t \in S$ with $\langle s, t\rangle$ finite, the set $\{s, t\}$ is conjugate to some $\{s', t'\} \subset S'$. (Setting $s = t$ shows that angle-compatible implies reflection-compatible.)

Mühlherr's Twist Conjecture predicts that angle-compatible Coxeter generating sets of a Coxeter group differ by a sequence of elementary twists. We postpone the definition of an elementary twist to give a brief historical background. For an exhaustive 2006 state of affairs, see [9].

The Isomorphism Problem for Coxeter groups asks for an algorithm to determine if Coxeter systems $(W, S), (W', S')$ defined by m_{st}, m'_{st} give rise to isomorphic groups W and W'. Hence listing all Coxeter generating sets S of W' solves the Isomorphism Problem. The articles of Howlett and Mühlherr [7], and Marquis and Mühlherr [8] reduce the question of listing all such sets S to the problem of listing all S angle-compatible with S'. In this way the Twist Conjecture describes a possible solution to the Isomorphism Problem for Coxeter groups.

The first substantial work on the Twist Conjecture is the one by Charney and Davis [4], where they prove that if a group acts effectively, properly, and cocompactly on a contractible manifold, then all its Coxeter generating sets are conjugate. Caprace and Mühlherr [2] proved that for all $m_{st} < \infty$, a Coxeter generating set S angle-compatible with S' is conjugate to S'. This is what was predicted by the Twist Conjecture, since S with all $m_{st} < \infty$ does not admit any elementary twist. Building on that, Caprace and Przytycki [3] proved that an arbitrary S not admitting any elementary twist, and angle-compatible with S', is in fact conjugate to S'. This should be considered as the "base case" of the Twist Conjecture.

In a foundational article [10] Mühlherr and Weidmann verified the Twist Conjecture in the case where all $m_{st} \geq 3$. In that case there occur twists in $\mathbb{Z}_2$ as

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well as in dihedral groups. Ratcliffe and Tschantz proved the Twist Conjecture for chordal Coxeter groups [12]. In these papers the assumptions on m_{st} seem an artefact of the proposed proof. In our paper, we propose the following "step one" of a systematic approach towards Twist Conjecture. Our first assumption below is natural from the point of view of the statement of the conjecture, since it says that the occurring elementary twists are as simple as possible. Our second assumption is that S is of *type* FC meaning that for any $T \subseteq S$ with m_{tr} finite for all $t, r \in T$, we have that $\langle T \rangle$ is finite. This assumption seems less natural from the point of view of the conjecture statement, but plays a role already in our proof of the "base case" [3].

Main Theorem. *Let S be a Coxeter generating set angle-compatible with S'. Suppose that S admits only twists in $\mathbb{Z}_2$, and is of type* FC*. Then S' is obtained from S by a sequence of elementary twists and a conjugation.*

We finally define an elementary twist. Let (W, S) be a Coxeter system. Given a subset $J \subseteq S$, we denote by W_J the subgroup of W generated by J. We call J *spherical* if W_J is finite. If J is spherical, let w_J denote the longest element of W_J. We say that two elements $s \neq t \in S$ are *adjacent* if $\{s, t\}$ is spherical. This gives rise to a graph whose vertices are S and whose edges (labelled by m_{st}) correspond to adjacent pairs of S. This graph is called the *defining graph* of S. Occasionally, when all m_{st} are finite, we will use another graph, whose vertices are still S, but (labelled) edges correspond to pairs of non-commuting elements of S. This graph is called the *Coxeter–Dynkin diagram* of S. Whenever we talk about adjacency of elements of S, we always mean adjacency in the defining graph unless otherwise specified.

Given a subset $J \subseteq S$, we denote by $J^\perp$ the set of those elements of $S \setminus J$ that commute with J. A subset $J \subseteq S$ is *irreducible* if it is not contained in $K \cup K^\perp$ for some non-empty proper subset $K \subset J$.

Let $J \subseteq S$ be an irreducible spherical subset. We say that $C \subseteq S \setminus (J \cup J^\perp)$ is a *component*, if the subgraph induced on C in the defining graph of S is a connected component of the subgraph induced on $S \setminus (J \cup J^\perp)$. Assume that we have a nontrivial partition $S \setminus (J \cup J^\perp) = A \sqcup B$, where each component C is contained entirely in A or in B. In other words, for all $a \in A$ and $b \in B$, we have that a and b are non-adjacent. We then say that J *weakly separates* S. In the language of groups, this means that W splits as an amalgamated product over $W_{J \cup J^\perp}$. Note that A and B are in general not uniquely determined by J.

We then consider the map $\tau \colon S \to W$ defined by

$$\tau(s) = \begin{cases} s & \text{for } s \in A \cup J \cup J^\perp, \\ w_J s w_J^{-1} & \text{for } s \in B, \end{cases}$$

which is called an *elementary twist in* $\langle J \rangle$ (see [1, Def 4.4]).

Coxeter generating sets S and S' of W are *twist equivalent* if S' can be obtained from S by a finite sequence of elementary twists and a conjugation. We say that S is *k-rigid* if for each weakly separating $J \subset S$ we have $|J| < k$. Thus 1-rigid means that there are no elementary twists (this was called *twist-rigid* in [3]). Our Main Theorem states that if a Coxeter generating set S is 2-rigid, of type FC, and angle-compatible to S', then it is twist equivalent to S'. Since twists in $\mathbb{Z}_2$ do not change the defining graph, it follows that S and S' have the same defining graphs. Note that right-angled Coxeter groups are 2-rigid, and that the Isomorphism Problem for these groups was solved by Radcliffe [11].

Organisation. In the entire article (except for Lemma 2.9) we assume that S **is irreducible, non-spherical, and of type FC**. (The reducible case easily follows from the irreducible.)

In Section 2 we recall some basic properties of the Davis complex, and several notions and results from [3]. In Section 3 we extend in two different ways a marking compatibility result from [3]. Section 4 contains a technical result required for the definition of complexity used in the proof of the Main Theorem in Section 5.

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2. Preliminaries

2.1. **Davis complex.** Let $\mathbb{A}$ be the *Davis complex* of a Coxeter system (W, S). The 1-skeleton of $\mathbb{A}$ is the Cayley graph of (W, S) with vertex set W and a single edge spanned on $\{w, ws\}$ for each $w \in W, s \in S$. Higher dimensional cells of $\mathbb{A}$ are spanned on left cosets in W of remaining finite W_J. The left action of W on itself extends to the action on $\mathbb{A}$.

A *chamber* is a vertex of $\mathbb{A}$. Collections of chambers corresponding to cosets wW_J are called *J-residues* of $\mathbb{A}$. A *gallery* is an edge-path in $\mathbb{A}$. For two chambers $c_1, c_2 \in \mathbb{A}$, we define their *gallery distance*, denoted by $d(c_1, c_2)$, to be the length of a shortest gallery from c_1 to c_2 (this coincides with the word-metric w.r.t. S).

Let $r \in W$ be an S-reflection. The fixed point set of the action of r on $\mathbb{A}$ is called its *wall* $\mathcal{W}_r$. The wall $\mathcal{W}_r$ determines r uniquely. Moreover, $\mathcal{W}_r$ separates $\mathbb{A}$ into two connected components, which are called *half-spaces (for r).* If a nonempty $K \subset \mathbb{A}$ is contained in a single half-space (this happens for example if K is connected and disjoint from $\mathcal{W}_r$), then $\Phi(\mathcal{W}_r, K)$ denotes this half-space. An edge of $\mathbb{A}$ crossed by $\mathcal{W}_r$ is *dual* to $\mathcal{W}_r$. A chamber is *incident* to $\mathcal{W}_r$ if it is an endpoint of an edge dual to $\mathcal{W}_r$. The *distance* of a chamber c to $\mathcal{W}_r$, denoted by $d(c, \mathcal{W}_r)$, is the minimal gallery distance from c to a chamber incident to $\mathcal{W}_r$.

The following fact is standard, see eg. [13, Thm 2.9].

Theorem 2.1. *Let $\mathcal{R}$ be a residue and let $x \in \mathcal{R}$ and $y \in W$ be chambers. Then there is a chamber $x' \in \mathcal{R}$ on a minimal length gallery from y to x such that $\Phi(\mathcal{W}_r, y) = \Phi(\mathcal{W}_r, x')$ for any reflection r fixing $\mathcal{R}$.*

2.2. **Bases and markings.** In this section we recall, in simplified form, several central notions from [3]. Let (W, S) be a Coxeter system. Let $\mathbb{A}_{\mathrm{ref}}$ be the Davis complex for (W, S) ("ref" stands for "reference complex"). For each reflection r, let $\mathcal{Y}_r$ be its wall in $\mathbb{A}_{\mathrm{ref}}$. The following was called *simple base with spherical support* in [3].

Definition 2.2. [3, Def 3.1 and 3.6] A *base* is a pair (s, w) with *core* $s \in S$ and $w \in W$ satisfying

(i) $w = j_1 \cdots j_n$ where j_i are pairwise distinct elements from $S \setminus \{s\}$,
(ii) $d(w.c_0, \mathcal{Y}_s) = n$,
(iii) every wall that separates $w.c_0$ from c_0 intersects $\mathcal{Y}_s$, and
(iv) the *support* $J = \{s, j_1, \ldots, j_n\}$ is spherical.

In [3, Lem 3.7] and the paragraph preceding it, we established the following.

Remark 2.3. (i) If $J \subset S$ is irreducible spherical and $s \in J$, then there exists a base with support J and core s. Namely, it suffices to order the elements of $J \setminus \{s\}$ into a sequence (j_i) so that for every $1 \leq i \leq n$ the set $\{s, j_1, \ldots, j_i\}$ is irreducible. Then $(s, j_1 \ldots j_n)$ is a base.

(ii) The core s and support J determine the base (s, w) uniquely. Hence we sometimes write a base as (s, J), or even just J if the core is understood. When $J = \{s\}$, we often write s instead of $\{s\}$ for simplicity.

Definition 2.4. A *marking* is a pair $\mu = ((s, J), m)$, where (s, J) is a base and where the *marker* $m \in S$ is not adjacent to some element of J. The *core* and the *support* of the marking μ are the core and the support of its base.

Our marking satisfies (but is not equivalent to the marking defined by) [3, Def 3.8]. To see that, note that by [3, Rem 3.12], we have that $w\mathcal{Y}_m$ is disjoint from $\mathcal{Y}_s$.

Remark 2.5. Let (s, J) be a base and $m \in S \setminus (J \cup J^{\perp})$. If $J \cup \{m\}$ is not spherical, then since S is of type FC, the pair $((s, J), m)$ is a marking. In particular, since S is irreducible non-spherical, we have that for each $s \in S$ there exists a marking with core s, since we can start with $J \subset S$ maximal irreducible spherical containing s. Similarly, for each $s \in I \subset S$ with I irreducible spherical, there exists a marking with core s and support containing I.

Now suppose that S is reflection-compatible with another Coxeter generating set S'. Let $\mathbb{A}_{\mathrm{amb}}$ be the Davis complex for (W, S') ("amb" stands for "ambient complex"). For each reflection r, let $\mathcal{W}_r$ be its wall in $\mathbb{A}_{\mathrm{amb}}$. The following picks up the geometry of the walls $\mathcal{W}_s$ for $s \in S$ inside the ambient complex for S'.

Definition 2.6. Let $\mu = ((s, w), m)$ be a marking. We define $\Phi_s^{\mu} = \Phi(\mathcal{W}_s, w\mathcal{W}_m)$, which is the half-space for s in $\mathbb{A}_{\mathrm{amb}}$ containing $w\mathcal{W}_m$.

2.3. **Geometric set of reflections.** Let $S, S', W, \mathbb{A}_{\mathrm{ref}}$ and $\mathbb{A}_{\mathrm{amb}}$ be as before, and assume that S **is angle-compatible with** S'. Let $P \subseteq S$.

Definition 2.7. Let $\{\Phi_p\}_{p \in P}$ be a collection of half-spaces of $\mathbb{A}_{\mathrm{amb}}$ for $p \in P$. The collection $\{\Phi_p\}_{p \in P}$ is 2-*geometric* if for any pair $p, r \in P$, the set $\Phi_p \cap \Phi_r \cap \mathbb{A}^{(0)}_{\mathrm{amb}}$ is a fundamental domain for the action of $\langle p, r \rangle$ on $\mathbb{A}^{(0)}_{\mathrm{amb}}$. The set P is 2-*geometric* if there exists a 2-geometric collection of half-spaces $\{\Phi_p\}_{p \in P}$.

Remark 2.8. By [2, Thm 4.2], if $\{\Phi_p\}_{p \in P}$ is 2-geometric, then after possibly replacing each Φ_p by opposite half-space, the collection $\{\Phi_p\}_{p \in P}$ is *geometric*, meaning that $F = \bigcap_{p \in P} \Phi_p \cap \mathbb{A}^{(0)}_{\mathrm{amb}}$ is nonempty. This justifies calling 2-geometric P *geometric* for simplicity. In fact, by [5] (see also [6, Thm 1.2] and [2, Fact 1.6]), if P is geometric, then F is a fundamental domain for the action of $\langle P \rangle$ on $\mathbb{A}^{(0)}_{\mathrm{amb}}$, and for each $p \in P$ there is a chamber in F incident to $\mathcal{W}_p$. In particular, if $P = S$, then S is conjugate to S'.

Note that since S is angle-compatible to S', every 2-element subset of S is geometric. However, this does not mean that S is 2-geometric. Nevertheless, for S spherical, it is easy to inductively choose 2-geometric Φ_s, and by Remark 2.8 we obtain the following.

Lemma 2.9. *If S is spherical, then it is conjugate to S'.*

Corollary 2.10. *Let $J \subset S$ be spherical. Then J is conjugate to a spherical $J' \subset S'$. In particular, J is geometric, and if it is irreducible, there exist exactly 2 fundamental domains F for the action of $\langle J \rangle$ on $\mathbb{A}^{(0)}_{\mathrm{amb}}$ as in Remark 2.8.*

Proof. Let $P \subset S$ be maximal spherical containing J. Then $\langle P \rangle$ is a maximal finite subgroup of W. By [1, Thm 1.9], we have that $\langle P \rangle$ is conjugate to $\langle P' \rangle$ for a maximal spherical $P' \subset S'$. Thus we can assume without loss of generality that $P = S$ and $P' = S'$. It now suffices to apply Lemma 2.9. □

Lemma 2.11. *Let $J \subset S$ be irreducible spherical, and let F be a fundamental domain for $\langle J \rangle$ in $\mathbb{A}^{(0)}_{\mathrm{amb}}$ guaranteed by Corollary 2.10. Let $s \in J$ and define $w \in W$ via $(s, w) = (s, J)$. Then we have $\Phi(\mathcal{W}_s, F) = \Phi(\mathcal{W}_s, w.F)$.*

Proof. First suppose $S = S'$. If $c_0 \in F$, then by Definition 2.2(ii) we have $\Phi(\mathcal{W}_s, c_0) = \Phi(\mathcal{W}_s, w.c_0)$, as desired. Otherwise, we have $w_J.c_0 \in F$. The half-spaces $\Phi(\mathcal{W}_s, w_J.c_0)$ and $\Phi(\mathcal{W}_s, ww_J.c_0)$ are opposite to $\Phi(\mathcal{W}_s, c_0)$ and $\Phi(\mathcal{W}_s, w.c_0)$, so they coincide as well.

If $S \neq S'$, then by Corollary 2.10 we have $gJg^{-1} = J'$, where J' is a spherical subset of S'. Then (gsg^{-1}, gwg^{-1}) is a base for S', and by the previous paragraph we have $\Phi(\mathcal{W}_{gsg^{-1}}, g.F) = \Phi(\mathcal{W}_{gsg^{-1}}, gw.F)$. Translating by g^{-1} we obtain the statement in the lemma. □

The next result is essentially [3, Prop 5.2]. Except for Lemma 2.9 this is the only place where we use angle-compatibility (instead of reflection-compatibility). Note that our markings are particular markings of [3], but the proof of [3, Prop 5.2] only uses such markings if S is of type FC.

Proposition 2.12. *Suppose that $P \subseteq S$ is irreducible and non-spherical. Let $p_1, p_2 \in P$. Suppose that for each $i = 1, 2$, any marking μ with core p_i and support and marker in P gives the same $\Phi_{p_i} = \Phi^{\mu}_{p_i}$. Then the pair $\{\Phi_{p_1}, \Phi_{p_2}\}$ is geometric.*

We summarise Remark 2.8 and Proposition 2.12 in the following.

Corollary 2.13. *If for each $s \in S$ any marking μ with core s gives rise to the same Φ^{μ}_s, then S is conjugate to S'.*

Also note that since S is of type FC, by [3, Lem 4.2 and Thm 4.5] a 1-rigid subset $P \subseteq S$ satisfies the hypothesis of Proposition 2.12.

Corollary 2.14. *If $P \subseteq S$ is 1-rigid, then it is geometric.*

3. Compatibility of markings

Let $S, S', W, \mathbb{A}_{\mathrm{ref}}$ and $\mathbb{A}_{\mathrm{amb}}$ be as in Section 2.3.

Definition 3.1. [3, Def 4.1] Let $((s, J), m), ((s, J'), m')$ be markings with common core. We say that they are related by *move*

(M1) if $J = J'$, and the markers m and m' are adjacent;
(M2) if there is $j \in S$ such that $J = J' \cup \{j\}$ and moreover m equals m' and is adjacent to j.

We will write $((s, J), m) \sim ((s, J'), m')$ if there is a finite sequence of moves of type M1 or M2 that brings $((s, J), m)$ to $((s, J'), m')$.

The following is a special case of [3, Lem 4.2].

Lemma 3.2. *If markings μ and μ' with common core s are related by move* M1 *or* M2*, then $\Phi_s^\mu = \Phi_s^{\mu'}$.*

The goal of this section is to provide two generalisations of [3, Thm 4.5].

Proposition 3.3. *Let $I \subset S$ be irreducible spherical. Suppose that no irreducible spherical $I' \supsetneq I$ weakly separates S. Let $\mu_1 = (J_1, m_1)$ and $\mu_2 = (J_2, m_2)$ be markings with common core $s \in I$ and such that $I \subseteq J_1, J_2$. Moreover, for $i = 1, 2$, define $K_i = J_i \setminus (I \cup I^\perp)$ when $I \subsetneq J_i$, and $K_i = \{m_i\}$ when $J_i = I$. Suppose that K_1 and K_2 are in the same component C of $S \setminus (I \cup I^\perp)$. Then $\mu_1 \sim \mu_2$. Consequently $\Phi_s^{\mu_1} = \Phi_s^{\mu_2}$.*

Proof. We follow the proof of Wojtaszczyk [3, App C], and argue by contradiction. Let I be maximal irreducible spherical satisfying the hypothesis of the proposition but with $\mu_1 \not\sim \mu_2$.

The *I-distance* between μ_1 and μ_2 is the length of a shortest edge-path in (the subgraph induced on) C between a vertex of K_1 and a vertex of K_2. (Such a path exists by our hypotheses.) Among pairs μ_1, μ_2 as above choose a pair with minimal I-distance.

If the I-distance between μ_1 and μ_2 is 0, then either $\{m_1\} = K_1 = K_2 = \{m_2\}$ yielding $\mu_1 = \mu_2$, which is a contradiction, or $J_1 \cap J_2 \setminus (I \cup I^\perp) \neq \emptyset$ giving a contradiction with the maximality of I.

Now assume that the I-distance between μ_1 and μ_2 is 1. Then there are two cases to consider. First consider the case where one of J_i, say J_1, equals I. If also $J_2 = I$, then m_1 and m_2 are adjacent. Thus μ_1 and μ_2 are related by move M1, which is a contradiction. If $I \subsetneq J_2$, then there exists $k_2 \in J_2 \setminus (I \cup I^\perp)$ such that k_2 and m_1 are adjacent. Thus μ_1 is related to $(I \cup \{k_2\}, m_1)$ by move M2. However, $(I \cup \{k_2\}, m_1) \sim \mu_2$ by the maximality of I, which is a contradiction. It remains to consider the case where $I \subsetneq J_1, J_2$. Then there exist $k_i \in J_i \setminus (I \cup I^\perp)$ such that k_1 and k_2 are adjacent. Note that $I \cup \{k_1, k_2\}$ is spherical and irreducible. By Remark 2.5, there exists a marking ν with core s and support containing $I \cup \{k_1, k_2\}$. By the maximality of I, we have $\mu_1 \sim \nu \sim \mu_2$, which is a contradiction.

If the I-distance between μ_1 and μ_2 is ≥ 2, let γ be a shortest edge-path in C connecting a vertex $k_1 \in K_1$ to a vertex $k_2 \in K_2$. Let k be the vertex on γ following k_1. If $I \cup \{k\}$ is spherical, then again by Remark 2.5, there exists a marking ν with core s and support containing $I \cup \{k\}$. Since we chose μ_1 and μ_2 to have minimal I-distance, we obtain $\mu_1 \sim \nu \sim \mu_2$, which is a contradiction. If $I \cup \{k\}$ is not spherical, then (I, k) is a marking, hence analogously $\mu_1 \sim (I, k) \sim \mu_2$, which is a contradiction. □

Proposition 3.4. *Let $P \subseteq S$ be irreducible non-spherical. Suppose that for any irreducible spherical $L \subset S$ with $L \cap P \neq \emptyset$, all elements of $P \setminus (L \cup L^\perp)$ are in one component of $S \setminus (L \cup L^\perp)$. Then for any markings μ_1 and μ_2 with supports and markers in P and common core p, we have $\mu_1 \sim \mu_2$. Consequently $\Phi_p^{\mu_1} = \Phi_p^{\mu_2}$ and by Proposition 2.12, P is geometric.*

Note that $P \setminus (L \cup L^\perp) \neq \emptyset$ for any irreducible spherical L. In the proof we will need the following terminology (depending on P). A marking $\mu = ((p, J), m)$ is *admissible* if

(1) $p \in P$, and

(2) if $L \subset S$ is irreducible spherical such that $p \in L$ and $J \nsubseteq L$, then $J \setminus (L \cup L^{\perp})$ (which is nonempty) and $P \setminus (L \cup L^{\perp})$ are in the same component of $S \setminus (L \cup L^{\perp})$, and
(3) if $L \subset S$ is irreducible spherical such that $J \subseteq L$, then m and $P \setminus (L \cup L^{\perp})$ are in the same component of $S \setminus (L \cup L^{\perp})$.

We also say that a base (p, J) is *admissible* if it satisfies Conditions (1) and (2).

Lemma 3.5. *Suppose that (p, J) is admissible. Let $\nu = ((p, J'), m)$ be such that $J \subseteq J'$, $J' \setminus J \subset P$ and $m \in P$. Then ν is admissible.*

Note that since P is irreducible non-spherical, such ν exists for each J.

Proof. Condition (1) is immediate. For Condition (2), pick irreducible spherical $L \subset S$ such that $p \in L$ and $J' \nsubseteq L$. If $J \nsubseteq L$, then $\emptyset \neq J \setminus (L \cup L^{\perp}) \subseteq J' \setminus (L \cup L^{\perp})$. Since (p, J) is admissible, Condition (2) holds for such L and J'. If $J \subseteq L$, then $J' \setminus (L \cup L^{\perp}) \subseteq J' \setminus J \subset P$, hence Condition (2) holds for such L and J'. Condition (3) is immediate, since we have $m \in P$. □

Proof of Proposition 3.4. It is clear that for $p \in P$ the base $(p, \{p\})$ is admissible. Thus by Lemma 3.5 both μ_1 and μ_2 are admissible. Hence to prove the proposition it suffices to show that for any two admissible markings μ_1, μ_2 with common core p, we have $\mu_1 \sim \mu_2$.

We argue by contradiction. Let $I \ni p$ be maximal irreducible spherical such that there are admissible markings $\mu_1 = (J_1, m_1)$ and $\mu_2 = (J_2, m_2)$ with $I \subseteq J_1, J_2$, and $\mu_1 \nsim \mu_2$. We define K_1, K_2, and the I-distance between μ_1 and μ_2 as in the proof of Proposition 3.3. Since both μ_1 and μ_2 are admissible, their I-distance is finite. Among pairs μ_1, μ_2 as above choose a pair with minimal I-distance.

If the I-distance is 0, then either $\mu_1 = \mu_2$, or there is irreducible $I' \supsetneq I$ contained in both J_1 and J_2, contradiction. Suppose now that the I-distance is 1. There are two cases to consider.

Case 1: $J_1 = I$. If $J_2 = I$, then μ_1 and μ_2 are related by move M1, contradiction. Now we assume $I \subsetneq J_2$. Pick $k_2 \in K_2$ adjacent to m_1. Then $I' = I \cup \{k_2\}$ is spherical and irreducible. Moreover, $\mu_1 \sim (I', m_1)$ by move M2. We claim that (I', m_1) is admissible. Then $(I', m_1) \sim \mu_2$ by the maximality of I, which yields a contradiction. Now we prove the claim. For Condition (2), let $p \in L$ and $I' \nsubseteq L$. If $I \nsubseteq L$, it suffices to use Condition (2) in the admissibility of μ_1. Now suppose $I \subseteq L$. Then $I' \setminus (L \cup L^{\perp}) = \{k_2\}$. By Condition (2) in the admissibility of μ_2, we have that k_2 is in the same component of $S \setminus (L \cup L^{\perp})$ as $P \setminus (L \cup L^{\perp})$, as desired. Condition (3) follows immediately from Condition (3) in the admissibility of μ_1.

Case 2: $I \subsetneq J_1$ **and** $I \subsetneq J_2$. For $i = 1, 2$, pick $k_i \in K_i$ such that k_1 and k_2 are adjacent. Then $J = I \cup \{k_1, k_2\}$ is spherical and irreducible. It is easy to show that J is admissible following the argument from Case 1. Let ν be an admissible marking constructed from J as in Lemma 3.5. Then $\mu_1 \sim \nu \sim \mu_2$ by the maximality of I, which yields a contradiction.

Finally suppose that the I-distance d between μ_1 and μ_2 is ≥ 2. Let γ be a shortest edge-path in the subgraph induced on $S \setminus (I \cup I^{\perp})$ starting at $k_1 \in K_1$ and ending at $k_2 \in K_2$. Let k be the vertex on γ following k_1. If $J = I \cup \{k\}$ is not spherical, then let $\nu = (I, k)$, otherwise let ν be defined from J as in Lemma 3.5. Since the I-distance between ν and μ_1, μ_2 is $< d$, to reach a contradiction it suffices to prove that ν is admissible.

Consider first the case where J is spherical. By Lemma 3.5, it suffices to prove that J is admissible. Let $p \in L$ and $J \subsetneq L$. If $I \subsetneq L$, then we use the admissibility of μ_1. Otherwise, we have $J \setminus (L \cup L^\perp) = \{k\}$. Since γ is a geodesic, $\gamma \cap L$ is empty, a vertex, or an edge. Thus there is a subpath of γ from k to k_1 or k_2 outside L. Since μ_1, μ_2 were admissible, k is in the component of $S \setminus (L \cup L^\perp)$ containing $P \setminus (L \cup L^\perp)$, as desired. The case where J is not spherical is similar. □

4. Relative position of maximal spherical subsets

In this section, we introduce particular subsets of pairs of maximal spherical residues (which will be in Section 5 involved in the definition of the complexity of a Coxeter generating set with respect to another one). It is crucial to prove that these subsets are well-defined (Proposition 4.3) which is the most technical part of the article, and we recommend to skip it at first reading. Let $S, S', W, \mathbb{A}_{\mathrm{ref}}$ and $\mathbb{A}_{\mathrm{amb}}$ be as in Section 2.3. **Throughout the remaining part of the article, we will also assume that S is 2-rigid**.

Let $J \subset S$ be a maximal spherical subset. By Corollary 2.10, W_J stabilises a unique maximal cell $\sigma_J \subset \mathbb{A}_{\mathrm{amb}}$. Let C_J be the collection of vertices in this maximal cell and let D_J be the elements of C_J incident to each $\mathcal{W}_j$ for $j \in J$. When J is irreducible, then by Corollary 2.10, it is easy to see that D_J is made of two antipodal vertices. In general, let $J = J_1 \sqcup \cdots \sqcup J_k$ be the decomposition of J into maximal irreducible subsets. Let $\sigma_J = \sigma_1 \times \cdots \times \sigma_k$ be the induced product decomposition of the associated cell. Then D_J is a product of pairs of antipodal vertices $\{u_i, v_i\}$ for each σ_i. Let $\pi_i \colon D_J \to \{u_i, v_i\}$ be the coordinate projections.

Definition 4.1. Let $J_1 \subset S$ be irreducible spherical and $r \in S$. A vertex $t \in J_1$ is *good with respect to* r, if t is adjacent to r, or $J_1 \setminus (t \cup t^\perp)$ is non-empty and in the same component of $S \setminus (t \cup t^\perp)$ as r. Note that being good depends on J_1.

Let J and I be two maximal spherical subsets of S. A maximal irreducible subset J_1 of J is *good* with respect to I if there exist non-adjacent $t \in J_1$ and $r \in I$ such that t is good with respect to r.

Definition 4.2. For each ordered pair (J, I) of maximal spherical subsets of S, we define the following subset $E_{J,I} \subseteq D_J$. First, for each i, consider the following $E^i_{J,I} \subseteq D_J$. If J_i is not good with respect to I, then we take $E^i_{J,I} = D_J$. If J_i is good, then let t and r be as in Definition 4.1. Then we take $E^i_{J,I} = D_J \cap \Phi(\mathcal{W}_t, \mathcal{W}_r)$ (which is $\pi_i^{-1}(u_i)$ or $\pi_i^{-1}(v_i)$). We define $E_{J,I} = E^1_{J,I} \cap \cdots \cap E^k_{J,I}$.

The goal of this section is to prove the following, saying that $E^i_{J,I}$ does not depend on the choice of t and r.

Proposition 4.3. *Let J_1, J and I be as Definition 4.1. Suppose that we have pairs of non-adjacent vertices (t, r) and (t', r') in $J_1 \times I$ such that t is good with respect to r, and t' is good with respect to r'. Then $E^1_{J,I} = {E^1_{J,I}}'$, where ${E^1_{J,I}}' = D_J \cap \Phi(\mathcal{W}_{t'}, \mathcal{W}_{r'})$.*

We need some preparatory lemmas.

Lemma 4.4. *Let $s, t \in S$ be adjacent, and let $r \in S$ be neither adjacent to s nor to t. Suppose that r and s are in distinct components of $S \setminus (t \cup t^\perp)$, and that r and t are in distinct components of $S \setminus (s \cup s^\perp)$ (in particular s and t do not commute). Let $J = \{s, t\}$. Then each point in (the unique component) $S \setminus (J \cup J^\perp)$ is neither adjacent to s nor to t.*

Proof. Suppose that the collection of vertices of $S \setminus (J \cup J^{\perp})$ that are adjacent to s or to t is non-empty. Since S is 2-rigid, there is a shortest edge-path γ in the subgraph induced on $S \setminus (J \cup J^{\perp})$ that connects r to a vertex $p \in S \setminus (J \cup J^{\perp})$ adjacent to s or t. We assume without loss of generality that p is adjacent to t. Since r and t are in distinct components of $S \setminus (s \cup s^{\perp})$, there is a vertex p' of γ in $s^{\perp}$. If $p \neq p'$, then the subpath $\gamma' \subseteq \gamma$ from r to p' is a shorter path from r to a vertex adjacent to s or t, which is a contradiction. If $p = p'$, then since r and s are in distinct components of $S \setminus (t \cup t^{\perp})$, there exists a vertex p'' of $\gamma' = \gamma$ in $t^{\perp}$. If $p'' \neq p$, then we can reach a contradiction as before. If $p'' = p$, then $p \in J^{\perp}$, which is impossible by our choice of γ. □

Lemma 4.5. *Let $t, r \in S$ be non-adjacent. Let $J \subset S$ be maximal spherical containing t and let J_1 be the maximal irreducible subset of J containing t. Let $j_0 \in J_1$ and let $\omega = (j_0, j_1, \ldots)$ be the geodesic edge-path in the Coxeter–Dynkin diagram of J_1 that starts at j_0 and ends at t (such a geodesic is unique since the Coxeter–Dynkin diagram of a spherical subset is a tree). Let j_n be the first vertex of ω not adjacent to r (possibly $j_n = j_0$ or $j_n = t$). Suppose that both $t, j_0 \in J_1$ are good w.r.t. r. Then we have $\Phi(j_n j_{n-1} \cdots j_1 \mathcal{W}_{j_0}, E^1_{J,I}) = \Phi(j_n j_{n-1} \cdots j_1 \mathcal{W}_{j_0}, \mathcal{W}_r)$.*

Proof. We write $E = E^1_{J,I}$ to shorten the notation.

We claim that for any non-commuting $j, j' \in J_1$ at least one of j, j' is good (w.r.t. r; we will skip repeating this in this proof). To justify the claim, if both j and j' are not good, then r and j are in distinct components of $S \setminus (j' \cup j'^{\perp})$, and r and j' are in distinct components of $S \setminus (j \cup j^{\perp})$. If $\{j, j'\} \subsetneq J_1$, then there is an element in $S \setminus (\{j, j'\} \cup \{j, j'\}^{\perp})$ adjacent to j and j', which contradicts Lemma 4.4. If $\{j, j'\} = J_1$, then one of j, j' equals t, which was assumed to be good, contradiction. This justifies the claim.

If $j_0 = t$, then there is nothing to prove. Otherwise, we induct on the length of ω and assume that the conclusion of the lemma holds for all good j_i distinct from j_0. By the claim either j_1 or j_2 is good. We look first at the situation where j_1 is good. There are four cases to consider.

Case 1: both j_0 and j_1 are not adjacent to r. Since both j_0 and j_1 are good, we deduce from Proposition 3.3 and the assumption that S is 2-rigid that $(j_1, r) \sim ((j_1, j_0), r)$ and $(j_0, r) \sim ((j_0, j_1), r)$. Let $\Sigma \subset \mathbb{A}_{\mathrm{amb}}$ be the union of the two sectors of the form $\Phi_{j_0} \cap \Phi_{j_1}$ for $\{\Phi_{j_0}, \Phi_{j_1}\}$ geometric. Then $(j_1, r) \sim ((j_1, j_0), r)$ implies $\mathcal{W}_r \subset \Sigma \cup j_0\Sigma$ and $(j_0, r) \sim ((j_0, j_1), r)$ implies $\mathcal{W}_r \subset \Sigma \cup j_1\Sigma$. Thus $\mathcal{W}_r \subset \Sigma$. By induction assumption, $\Phi(\mathcal{W}_{j_1}, E) = \Phi(\mathcal{W}_{j_1}, \mathcal{W}_r)$, thus E and $\mathcal{W}_r$ are in the same sector of Σ, and it follows that $\Phi(\mathcal{W}_{j_0}, E) = \Phi(\mathcal{W}_{j_0}, \mathcal{W}_r)$.

Case 2: j_1 is adjacent to r, but j_0 is not adjacent to r. Then $n = 0$. Let j_m be the first vertex of ω distinct from j_0 not adjacent to r.

First, we claim $\Phi(j_0 \mathcal{W}_{j_1}, E) = \Phi(j_0 \mathcal{W}_{j_1}, \mathcal{W}_r)$. Indeed, since (j_1, j_0) and $(j_1, j_2 \cdots j_m)$ are bases, by two applications of Lemma 2.11 we have

$$\Phi(\mathcal{W}_{j_1}, j_0.E) = \Phi(\mathcal{W}_{j_1}, E) = \Phi(\mathcal{W}_{j_1}, j_2 \cdots j_m.E),$$

which equals $\Phi(\mathcal{W}_{j_1}, j_2 \cdots j_m \mathcal{W}_r)$ by induction. Furthermore, $((j_1, j_2 \cdots j_m), r)$ is a marking and j_2 is adjacent to j_0. Thus by Proposition 3.3 and the fact that S is 2-rigid, we obtain

$$((j_1, j_2 \cdots j_m), r) \sim ((j_1, j_0), r),$$

and the claim follows.

Let Φ_{j_0}, Φ_{j_1} be the half-spaces for j_0, j_1 containing E and let $\Lambda = \Phi_{j_0} \cap \Phi_{j_1}$. Since $\mathcal{W}_r$ intersects $\mathcal{W}_{j_1}$, by the claim we have that $\mathcal{W}_r$ intersects Λ. It follows that $\Phi(\mathcal{W}_{j_0}, E) = \Phi(\mathcal{W}_{j_0}, \mathcal{W}_r)$.

Case 3: j_0 is adjacent to r, but j_1 is not adjacent to r. By induction, we have $\Phi(\mathcal{W}_{j_1}, E) = \Phi(\mathcal{W}_{j_1}, \mathcal{W}_r)$. We need to show $\Phi(j_1\mathcal{W}_{j_0}, E) = \Phi(j_1\mathcal{W}_{j_0}, \mathcal{W}_r)$. To do this, it suffices to reverse the argument in the previous paragraph.

Case 4: both j_0 and j_1 are adjacent to r. Let $P = \{j_0, j_1, \ldots, j_n, r\}$. We claim that P is geometric. Indeed, since P is irreducible and non-spherical, to justify the claim it suffices to verify the hypothesis of Proposition 3.4. Let $L \subset S$ be irreducible spherical with $L \cap P \neq \emptyset$. Since S is 2-rigid, it suffices to consider $L = \{l\}$ a singleton in P. Note that in P the only two non-adjacent elements are r and j_n. Thus the cases $l = r, j_n$ are clear. It remains to consider the case $l \in K = P \setminus \{r, j_n\}$. Since K is irreducible and $|K| \geq 2$, we have $K \setminus (l \cup l^\perp) \neq \emptyset$. Consequently, l does not weakly separate P, verifying the claim.

By Remark 2.8, there are half-spaces $\{\Phi_{j_0}, \Phi_{j_1}, \cdots, \Phi_{j_n}, \Phi_r\}$ whose intersection contains a vertex x incident to $\mathcal{W}_r$. Thus by induction we have

$$\Phi(j_n j_{n-1} \cdots j_2 \mathcal{W}_{j_1}, E) = \Phi(j_n j_{n-1} \cdots j_2 \mathcal{W}_{j_1}, \mathcal{W}_r) = \Phi(j_n j_{n-1} \cdots j_2 \mathcal{W}_{j_1}, x).$$

Let F and F_{ant} (resp. V and V_{ant}) be the two fundamental domains for $\{j_0, j_1, \ldots, j_n\}$ (resp. $\{j_1, \ldots, j_n\}$) from Corollary 2.10. Assume without loss of generality $F \subset V$. Then x and E are both inside F or F_{ant}, say F, otherwise they would be separated by $j_n j_{n-1} \cdots j_2 \mathcal{W}_{j_1}$. It follows that both x and E are in V. In particular, $\Phi(j_n j_{n-1} \cdots j_1 \mathcal{W}_{j_0}, E) = \Phi(j_n j_{n-1} \cdots j_1 \mathcal{W}_{j_0}, x)$, which equals $\Phi(j_n j_{n-1} \cdots j_1 \mathcal{W}_{j_0}, \mathcal{W}_r)$, as desired.

Now we turn to the situation where j_1 is not good, hence j_2 is good. Since j_1 is not good, it is not adjacent to r, and furthermore r is not adjacent to j_0 or j_2. Since j_0 is good and S is 2-rigid, by Proposition 3.3 we obtain $\Phi(\mathcal{W}_{j_0}, \mathcal{W}_r) = \Phi(\mathcal{W}_{j_0}, j_1\mathcal{W}_r) = \Phi(\mathcal{W}_{j_0}, j_1 j_2 \mathcal{W}_r)$. Similarly, $\Phi(\mathcal{W}_{j_2}, \mathcal{W}_r) = \Phi(\mathcal{W}_{j_2}, j_1\mathcal{W}_r) = \Phi(\mathcal{W}_{j_2}, j_1 j_0 \mathcal{W}_r)$. Since $\{j_0, j_1, j_2\}$ is conjugate to a subset of S' by Corollary 2.10, we deduce that $\Phi(\mathcal{W}_{j_0}, E) = \Phi(\mathcal{W}_{j_0}, \mathcal{W}_r)$ by applying Lemma 4.6 below with $s_1 = j_0$, $s_2 = j_1$ and $s_3 = j_2$. □

Lemma 4.6. *Let H be a 3-generator irreducible spherical Coxeter group and let σ be the associated Davis cell. Let s_1, s_2, s_3 be consecutive vertices in the Coxeter–Dynkin diagram of H, and let $\mathcal{W}_{s_1}$, $\mathcal{W}_{s_2}$ and $\mathcal{W}_{s_3}$ be the associated walls of σ. Let c_0 be a chamber of σ that is incident to each of $\mathcal{W}_{s_i}$ for $1 \leq i \leq 3$. Let c be an arbitrary chamber satisfying all of the following.*

(1) $\Phi(\mathcal{W}_{s_1}, c) = \Phi(\mathcal{W}_{s_1}, c_0)$;
(2) $\Phi(\mathcal{W}_{s_1}, c) = \Phi(\mathcal{W}_{s_1}, s_2 c) = \Phi(\mathcal{W}_{s_1}, s_2 s_3 c)$;
(3) $\Phi(\mathcal{W}_{s_3}, c) = \Phi(\mathcal{W}_{s_3}, s_2 c) = \Phi(\mathcal{W}_{s_3}, s_2 s_1 c)$.

Then $\Phi(\mathcal{W}_{s_3}, c) = \Phi(\mathcal{W}_{s_3}, c_0)$.

Proof. We will prove that either $c = c_0$, or $c = s_2.c_0$, which implies immediately the lemma. We first consider the case where H is the $(2,3,5)$-triangle group. Assume first $(s_2 s_3)^5 = 1$. Consider the tilling of the regular dodecahedron obtained from drawing all the walls of H. A direct computation gives Figure 1. It follows from Conditions (1) and (2) that $c \in \Phi(\mathcal{W}_{s_1}, c_0) \cap s_2\Phi(\mathcal{W}_{s_1}, c_0) \cap s_3 s_2 \Phi(\mathcal{W}_{s_1}, c_0)$. In other words, c is inside the region R_1 bounded by $\mathcal{W}_{s_1}$, $s_2\mathcal{W}_{s_1}$ and $s_3 s_2 \mathcal{W}_{s_1}$ containing c_0.

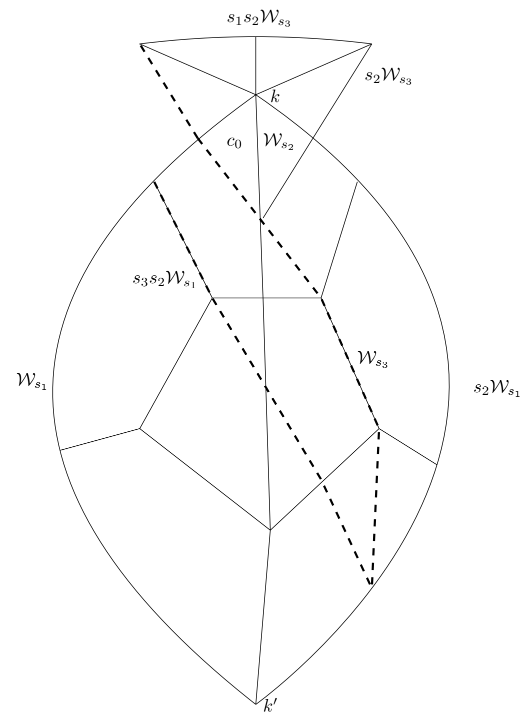
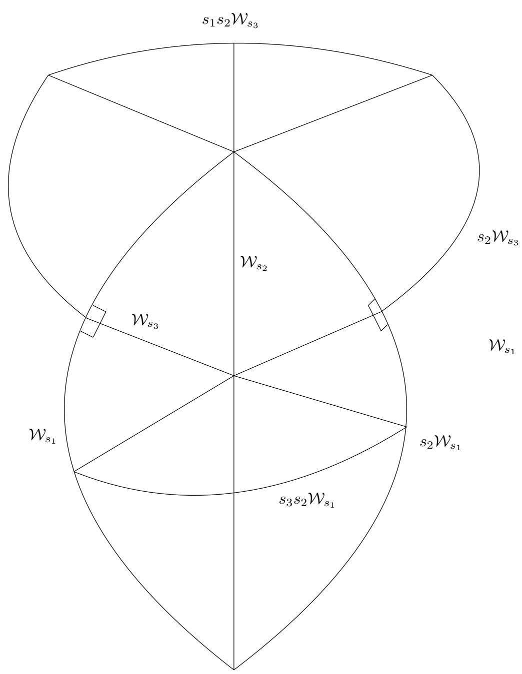
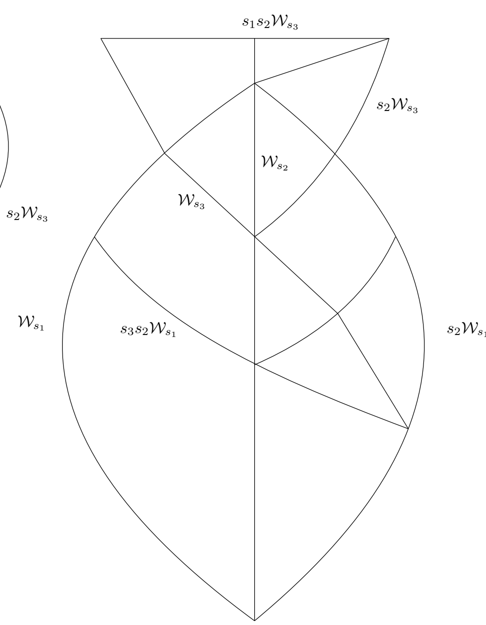


FIGURE 1. $(2,3,5)$-triangle group



(A) $(2,3,3)$



(B) $(2,3,4)$

Similarly, by Condition (3), c is inside the region R_2 bounded by $\mathcal{W}_{s_3}$, $s_2\mathcal{W}_{s_3}$ and $s_1s_2\mathcal{W}_{s_3}$ that contains either c_0 or its antipodal chamber. However, the latter case is impossible since $R_1 \cap R_2$ contains c and is thus non-empty. Thus $R_1 \cap R_2$ is the union of the two triangles adjacent along $\mathcal{W}_{s_2}$, one of which contains c_0, as desired.

If $(s_2s_3)^3 = 1$, the proof is analogous. The case of the $(2,3,3)$-triangle group and the $(2,3,4)$-triangle group can be proved in a similar way, see Figures 2a and 2b to chase down the relevant regions. □

We are finally ready for the following.

Proof of Proposition 4.3. We prove the proposition by induction on the distance between t and t' in the Coxeter–Dynkin diagram of J_1. If $t = t'$, then since $\mathcal{W}_r \cap \mathcal{W}_{r'} \neq \emptyset$, the proposition is clear. Also note that if $r = r'$, then the proposition follows from Lemma 4.5.

Now we assume $t \neq t'$ and $r \neq r'$. If t and r' are non-adjacent, then t is good with respect to r' (since r and r' are adjacent). Thus we can pass from (t, r) to (t', r') via (t, r') by the previous discussion. The case where t' and r are non-adjacent is analogous. Thus it remains to consider the case where t and r' are adjacent, and t' and r are adjacent.

We first look at the distance one case: where t and t' do not commute. We consider $P = \{t, t', r, r'\}$. Note that the defining graph of P is a square, thus P is 1-rigid. Hence P is geometric by Corollary 2.14. Let $F \subset \mathbb{A}^{(0)}_{\text{amb}}$ be the fundamental domain for $\langle P\rangle \curvearrowright \mathbb{A}^{(0)}_{\text{amb}}$ from Remark 2.8. Let $V \subset \mathbb{A}^{(0)}_{\text{amb}}$ be the fundamental domain for $\langle t, t'\rangle$ that contains F. Since t and t' do not commute, V is the only fundamental domain for $\langle t, t'\rangle$ contained in $\Phi(\mathcal{W}_t, \mathcal{W}_r)$ and the only one in $\Phi(\mathcal{W}_{t'}, \mathcal{W}_{r'})$. Thus $E^1_{J,I} \subset V$ and ${E^1_{J,I}}' \subset V$. It follows that $E^1_{J,I} = {E^1_{J,I}}'$.

Now we deal with the general situation. We consider the geodesic edge-path $(t_i)_{i=0}^n$ from $t_0 = t$ to $t_n = t'$ in the Coxeter–Dynkin diagram of J_1 (which is a tree). Let i' be minimal such that $t_{i'}$ is not adjacent to r' and i maximal such that t_i is not adjacent to r. Then $t_{i'}$ is good respect to r' and t_i is good with respect to r. Note that $i' \geq 1$ and $i \leq n-1$. If $i' \leq n-1$, then by the induction assumption we can pass from (t, r) to (t', r') via $(t_{i'}, r')$. The case $i \geq 1$ is analogous. Thus in the remaining part of the proof we assume $i' = n$ and $i = 0$, in other words, t_i is adjacent to both r and r' for each $1 \leq i \leq n-1$.

Let $P = \{t_0, \ldots, t_n, r, r'\}$. Note that the defining graph of P is a join of a 4-cycle (whose consecutive vertices are t, r', r, t') and a complete graph (whose vertices are $t_1, \ldots, t_{n-1}$). Since (t_i) was an edge-path in the Coxeter–Dynkin diagram, it is easy to prove that the defining graph of P is 1-rigid. Thus P is geometric by Corollary 2.14. Let $F \subset \mathbb{A}^{(0)}_{\text{amb}}$ be the fundamental domain for $\langle P\rangle \curvearrowright \mathbb{A}^{(0)}_{\text{amb}}$ from Remark 2.8. Let $V \subset \mathbb{A}^{(0)}_{\text{amb}}$ be the fundamental domain for $\langle t_0, \ldots, t_n\rangle$ that contains F. Since $\{t_0, \ldots, t_n\}$ is irreducible, V is the only fundamental domain for $\langle t_0, \ldots, t_n\rangle$ contained in $\Phi(\mathcal{W}_t, \mathcal{W}_r)$ and the only one in $\Phi(\mathcal{W}_{t'}, \mathcal{W}_{r'})$. Thus $E^1_{J,I} \subset V$ and ${E^1_{J,I}}' \subset V$. Hence $E^1_{J,I} = {E^1_{J,I}}'$. □

5. Proof of the main theorem

We keep the setup from Section 4.

Definition 5.1. We define the *complexity* of S, denoted $\mathcal{K}(S)$, to be an ordered pair of numbers

$$\big(\mathcal{K}_1(S), \mathcal{K}_2(S)\big) = \Big(\sum_{J\neq I} d(C_J, C_I), \sum_{J\neq I} d(E_{J,I}, E_{I,J})\Big),$$

where J and I range over all maximal spherical subsets of S, and $E_{J,I}$ is defined in Definition 4.2. Note that the distance is computed in $\mathbb{A}_{\mathrm{amb}}$ and so we have $\mathcal{K}_1(S') = \mathcal{K}_2(S') = 0$.

For two Coxeter generating sets S and S_τ, we define $\mathcal{K}(S_\tau) < \mathcal{K}(S)$ if $\mathcal{K}_1(S_\tau) < \mathcal{K}_1(S)$, or $\mathcal{K}_1(S_\tau) = \mathcal{K}_1(S)$ and $\mathcal{K}_2(S_\tau) < \mathcal{K}_2(S)$.

Note that since S is 2-rigid, an elementary twist does not change its defining graph. Thus Main Theorem reduces to the following.

Theorem 5.2. *Let S be angle-compatible with S'. Suppose that S is 2-rigid and of type* FC*. Assume moreover that S has minimal complexity among all Coxeter generating sets twist-equivalent to S. Then S is conjugate to S'.*

The proof will take the remaining part of the article, and we divide it into several steps. For $\mu = ((s,w),m)$ a marking with support J, we define $K_\mu = J \setminus \{s\}$ if $J \neq \{s\}$, and $K_\mu = \{m\}$ otherwise.

By Corollary 2.13, to prove Theorem 5.2 it suffices to show that for any markings μ and μ' with common core $s \in S$, we have $\Phi_s^\mu = \Phi_s^{\mu'}$. Note that for each component A of $S \setminus (s \cup s^\perp)$, there exists a marking μ with $K_\mu \subseteq A$. By Proposition 3.3 and the fact that S is 2-rigid, if $K_{\mu'} \subseteq A$, then $\Phi_s^\mu = \Phi_s^{\mu'}$. Thus each component A of $S \setminus (s \cup s^\perp)$ determines a half-space $\Phi_A := \Phi_s^\mu$ for s. Two components A_1 and A_2 of $S \setminus (s \cup s^\perp)$ are *compatible* if $\Phi_{A_1} = \Phi_{A_2}$. We will show that all the components of $S \setminus (s \cup s^\perp)$ are compatible. Fixing $s \in S$, we shall divide these components into several classes and conduct a case analysis.

5.1. **Big components are compatible.**

Definition 5.3. A component A of $S \setminus (s \cup s^\perp)$ is *big* if there is $a \in A$ not adjacent to s. Otherwise A is *small*.

Lemma 5.4. *Any two big components are compatible.*

Proof. We argue by contradiction and assume that the big components of $S \setminus (s \cup s^\perp)$ can be divided into two non-empty families $\{A_k\}$ and $\{B_k\}$ such that all Φ_{A_k} coincide (call that half-space Φ_A) and are distinct from all Φ_{B_k}, which also coincide (call that half-space Φ_B). Let B be the union of all the B_k. Let τ be the elementary twist that sends each element $b \in B$ to sbs and fixes other elements of S. For a contradiction, we will prove $\mathcal{K}_1(\tau(S)) < \mathcal{K}_1(S)$.

Let $J \subset S$ be maximal spherical. J is *twisted* if it contains an element of B and $s \notin J$. A twisted J exists, since we can take any maximal spherical J containing $b \in B$ not adjacent to s. Note that if J is twisted, then for each $j \in J$ we have $\mathcal{W}_{\tau(j)} = s\mathcal{W}_j$, and hence $C_{\tau(J)} = s.C_J$. Moreover, there is an element $b \in J \setminus \{s\}$ not adjacent to s, since otherwise $J \cup \{s\}$ would be spherical contradicting the maximality of J. Then $\Phi(\mathcal{W}_s, C_J) = \Phi(\mathcal{W}_s, \mathcal{W}_b) = \Phi_B$.

Consider now maximal spherical $I \subset S$ that is not twisted. If $s \in I$, then $C_{\tau(I)} = s.C_I = C_I$. If $s \notin I$, then $I \cap B = \emptyset$, and we also have $C_{\tau(I)} = C_I$. As before, there exists such I with $s \notin I$. Moreover, then there is $a \in I \setminus \{s\}$ not adjacent to s, and $\Phi(\mathcal{W}_s, C_I) = \Phi(\mathcal{W}_s, \mathcal{W}_a) = \Phi_A$.

Let $J, I \subset S$ be maximal spherical. If both J and I are twisted or both are not twisted, then $d(C_J, C_I) = d(C_{\tau(J)}, C_{\tau(I)})$. Now suppose that J is twisted and I is not twisted. If $s \in I$, we still have $d(C_J, C_I) = d(C_{\tau(J)}, C_{\tau(I)})$. If $s \notin I$, then since

$\Phi_B \neq \Phi_A$, we have $\Phi(\mathcal{W}_s, C_J) \neq \Phi(\mathcal{W}_s, C_I)$. Hence a minimal length gallery β from a chamber in C_J to a chamber in C_I has an edge dual to $\mathcal{W}_s$. Removing this edge from β and reflecting $\beta \cap \Phi(\mathcal{W}_s, C_J)$ by s, we obtain a shorter gallery from a chamber in $s.C_J$ to a chamber in C_I. Thus $d(C_{\tau(J)}, C_{\tau(I)}) = d(s.C_J, C_I) < d(C_J, C_I)$. Consequently $\mathcal{K}_1(\tau(S)) < \mathcal{K}_1(S)$. □

5.2. **Exposed components.**

Definition 5.5. A small component A is *exposed* if there is $t \in A$ and r inside a different component of $S \setminus (s \cup s^\perp)$ such that s and r are in distinct components of $S \setminus (t \cup t^\perp)$.

Lemma 5.6. *If there exists an exposed component, then all components are compatible.*

Proof. Let t and r be as in Definition 5.5. Note that r is adjacent to neither s nor t. By Lemma 4.4, none of the elements of $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ is adjacent to s or t. It follows that there is only one small component of $S \setminus (s \cup s^\perp)$, and this small component equals $\{t\}$.

Observe that a maximal spherical subset $J \subset S$ contains s if and only if it contains t. Indeed, if say $s \in J$, then each element of $J \setminus \{s\}$ is adjacent to s. Hence $J \subseteq \{s,t\} \cup \{s,t\}^\perp$ by Lemma 4.4. If $t \notin J$, then $J \cup \{t\}$ is spherical, which contradicts the maximality of J. We say that J is *exposed* if $\{s,t\} \subseteq J$.

Let $\mathcal{W}_{\{s,t\}}$ be the union of all the walls in $\mathbb{A}_{\mathrm{amb}}$ for the reflections in the dihedral group $\langle s,t \rangle$. Since S is 2-rigid, the graph induced on $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ is connected. Thus all the walls $\mathcal{W}_r$ for $r \in S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ lie in the same connected component Λ of $\mathbb{A}_{\mathrm{amb}} \setminus \mathcal{W}_{\{s,t\}}$. Consequently, all D_J for J not exposed lie in Λ. Let $\Sigma \subset \mathbb{A}_{\mathrm{amb}}$ be the union of the two sectors of the form $\Phi_s \cap \Phi_t$ for $\{\Phi_s, \Phi_t\}$ geometric. Assume first $\Lambda \subset \Sigma$. Then $\Phi(\mathcal{W}_s, \Lambda) = \Phi(\mathcal{W}_s, t\Lambda)$, hence $\Phi(\mathcal{W}_s, \mathcal{W}_r) = \Phi(\mathcal{W}_s, t\mathcal{W}_r)$. These half-spaces correspond to markings $\mu = ((s,t), r)$ with $K_\mu = \{t\}$ and $\mu' = (s,r)$ with $K_{\mu'} = \{r\}$. Consequently, the unique small component $\{t\}$ of $S \setminus (s \cup s^\perp)$ is compatible with a big component. In view of Lemma 5.4, all the components are compatible. It remains to consider the case $\Lambda \not\subset \Sigma$.

Let τ_s (resp. τ_t) be the elementary twist that sends t to sts (resp. s to tst) and fixes other elements of S. For any $w \in \langle s,t \rangle$, composing appropriately τ_s and τ_t (while keeping the notation s,t for the images of s,t under the twist), we obtain $\tau = \tau(w)$ sending s to wsw^{-1}, t to wtw^{-1} and fixing other elements of S. We will justify the following.

1. $\mathcal{W}_{\tau(s)} = w\mathcal{W}_s$ and $\mathcal{W}_{\tau(t)} = w\mathcal{W}_t$;
2. if J is maximal spherical that is exposed (resp. not exposed), then $D_{\tau(J)} = w.D_J$ (resp. $D_{\tau(J)} = D_J$);
3. if J and I are both exposed (resp. not exposed), then $E_{\tau(J),\tau(I)} = w.E_{J,I}$ (resp. $E_{\tau(J),\tau(I)} = E_{J,I}$);
4. if J is exposed and I is not exposed, then $E_{\tau(J),\tau(I)} = w.E_{J,I}$ and $E_{\tau(I),\tau(J)} = E_{I,J}$.

Here (1) is immediate and implies (2), while (3) follows from (2) and Definition 4.2 (note that an elementary twist does change the defining graph, so it does not change the good subsets of J and I). Now we prove (4). Note that for each $j \in J$, we have $\mathcal{W}_j \cap \mathcal{W}_{\tau(j)} \neq \emptyset$. Moreover, τ fixes each element of I. Thus for non-adjacent $i \in I$ and $j \in J$, the walls $\mathcal{W}_j$ and $\mathcal{W}_{\tau(j)}$ are in the same half-space for $i = \tau(i)$. Hence it follows from Definition 4.2 that $E_{\tau(I),\tau(J)} = E_{I,J}$. It remains to verify the first equality of (4). Note that the elements of $J \setminus \{s,t\}$ are fixed by τ, and $\{s,t\} \subset J$ is maximal irreducible that is not good in view of Definition 5.5 and Lemma 4.4. Thus $E_{\tau(J),\tau(I)} = D_{\tau(J)} = w.D_J = w.E_{J,I}$, finishing the proof of (4).

Coming back to the case $\Lambda \not\subset \Sigma$, choose $\tau = \tau(w)$ a composition of twists as above so that $w\Sigma$ contains Λ. We will reach a contradiction by showing $\mathcal{K}_1(\tau(S)) = \mathcal{K}_1(S)$ and $\mathcal{K}_2(\tau(S)) < \mathcal{K}_2(S)$. The equality follows from the fact that for any maximal spherical $J \subset S$, we have $C_{\tau(J)} = C_J$. Now we verify the inequality. Consider maximal spherical subsets $J, I \subset S$. If both J and I are exposed or both are not exposed, then by (3) we have $d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) = d(E_{J,I}, E_{I,J})$.

Now we assume that J is exposed but I is not exposed. Let β be a shortest gallery from a chamber $y \in E_{I,J}$ to a chamber $x \in E_{J,I}$. By angle-compatibility, $\{s,t\}$ is conjugate to $\{s',t'\} \subset S'$. By Theorem 2.1, we can assume that β is a concatenation of galleries β' and β'', where β' is a minimal gallery from y to some chamber (call it x') in the $\{s',t'\}$-residue $\mathcal{R}$ containing x. Furthermore, $\beta' \subset \Lambda$. Note that $x \neq x'$ since $\Lambda \nsubseteq \Sigma$.

We have $x' = w.x$ or $x' = w.x_{\mathrm{ant}}$, where x_{ant} is the chamber antipodal to x in $\mathcal{R}$. Note that $x_{\mathrm{ant}} \in E_{J,I}$, since $\{s,t\}$ is an irreducible component of J that is not good with respect to I. Thus from (4) we deduce $x' \in E_{\tau(J),\tau(I)}$ and $y \in E_{\tau(I),\tau(J)}$. Consequently $d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) < d(E_{J,I}, E_{I,J})$, giving $\mathcal{K}_2(\tau(S)) < \mathcal{K}_2(S)$. □

5.3. **Non-exposed small components.** To prove Theorem 5.2, it remains to consider the case where all components of $S \setminus (s \cup s^{\perp})$ are big, or small and not exposed. We argue by contradiction and assume that the components of $S \setminus (s \cup s^{\perp})$ can be divided into two non-empty families $\{A_k\}$ and $\{B_k\}$ such that all Φ_{A_k} coincide and are distinct from all Φ_{B_k}, which also coincide. Let A (resp. B) be the union of all B_k (resp. A_k). By Lemma 5.4, we can assume that all the big components (if they exist) are in A. Let τ be the elementary twist that sends each element $b \in B$ to sbs and fixes other elements of S. By the proof of Lemma 5.4, we have $\mathcal{K}_1(S) = \mathcal{K}_1(\tau(S))$. For a contradiction, we will prove $\mathcal{K}_2(\tau(S)) < \mathcal{K}_2(S)$.

Let $J \subset S$ be a maximal spherical subset. J is *twisted* if it contains an element of B. In that case, s is adjacent to each element in J since B is a union of small components. Consequently $J \cup \{s\}$ is spherical so $s \in J$ by the maximality of J.

Consider maximal spherical subsets J and I. If both of them are twisted or both are not-twisted, then we have

$$d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) = d(E_{J,I}, E_{I,J}). \tag{5.1}$$

Now we assume that J is twisted and I is not twisted. If $I \subseteq \{s\} \cup \{s\}^{\perp}$, then (5.1) holds as well. It remains to discuss the case where $I \nsubseteq s \cup s^{\perp}$. We will prove $d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) < d(E_{J,I}, E_{I,J})$, which implies $\mathcal{K}_2(\tau(S_0)) < \mathcal{K}_2(S_0)$ and finishes the proof of Theorem 5.2.

Case 1: I **contains** s. In that case, pick $r \in I \setminus (s \cup s^{\perp})$. Let $I_1 \subseteq I$ be maximal irreducible containing r. Then $s \in I_1$, since s and r do not commute. Pick $t \in J \setminus (s \cup s^{\perp})$. Let $J_1 \subseteq J$ be maximal irreducible containing t. Then $s \in J_1$. Since both t and r are adjacent to s, we have that $t \in J_1$ is good with respect to r, and $r \in I_1$ is good respect to t.

We first justify that $E_{J,I}$ and $E_{I,J}$ lie in distinct half-spaces for s. Otherwise, $\{r,s,t\}$ is geometric. In particular, we have $\Phi(\mathcal{W}_s, t\mathcal{W}_r) = \Phi(\mathcal{W}_s, r\mathcal{W}_t)$. These half-spaces correspond to markings $\mu = ((s,t),r)$ with $K_\mu = \{t\}$ and $\mu' = ((s,r),t)$ with $K_{\mu'} = \{r\}$. This contradicts the assumption that t and r belong to incompatible components.

We have $D_{\tau(J)} = s.D_J$. Note that τ fixes all the elements of I and $J \setminus J_1$, and hence $E_{\tau(J),\tau(I)} = s.E_{J,I}$ in view of

$$\Phi(s\mathcal{W}_t, \mathcal{W}_r) = \Phi(s\mathcal{W}_t, \mathcal{W}_r \cap \mathcal{W}_s) = s\Phi(\mathcal{W}_t, \mathcal{W}_r \cap \mathcal{W}_s) = s\Phi(\mathcal{W}_t, \mathcal{W}_r).$$

On the other hand, we have $E_{\tau(I),\tau(J)} = E_{I,J}$, since $\mathcal{W}_j \cap \mathcal{W}_{\tau(j)} \neq \emptyset$ for each $j \in J$, and hence $\mathcal{W}_j$ and $\mathcal{W}_{\tau(j)}$ are in the same half-space for $i = \tau(i) \in I$ not adjacent to j.

To conclude Case 1, pick a gallery β of minimal length from $x \in E_{J,I}$ to $y \in E_{I,J}$. Since chambers x and y lie in distinct half-spaces for s and x is incident to $\mathcal{W}_s$, we can assume that the first edge of β is dual to $\mathcal{W}_s$ (Theorem 2.1). Since $s.x \in s.E_{J,I} = E_{\tau(J),\tau(I)}$ and $y \in E_{I,J} = E_{\tau(I),\tau(J)}$, we have $d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) < d(E_{J,I}, E_{I,J})$, as desired.

Case 2: I contains an element not adjacent to s. Let this element be r. Let t and J_1 be as in Case 1. Since t is inside a non-exposed small component, $t \in J_1$ is good with respect to r. In particular, J_1 is good with respect to I.

Let $\Sigma \subset \mathbb{A}_{\mathrm{amb}}$ be the union of the two sectors of the form $\Phi_s \cap \Phi_t$ for $\{\Phi_s, \Phi_t\}$ geometric. We first justify $\mathcal{W}_r \subset s\Sigma$. Indeed, note that $\mathcal{W}_r$ is disjoint from any wall in $\mathcal{W}_{\{s,t\}}$. Since s and r are in the same component of $S \setminus (t \cup t^\perp)$, we have $(t,r) \sim ((t,s),r)$ by Proposition 3.3 and the fact that S is 2-rigid. Thus $\Phi(\mathcal{W}_t, \mathcal{W}_r) = \Phi(\mathcal{W}_t, s\mathcal{W}_r)$. It follows that $\mathcal{W}_r \subset \Sigma \cup s\Sigma$. Now recall that $t \in B$ and $r \in A$, thus $\Phi(\mathcal{W}_s, \mathcal{W}_r) \neq \Phi(\mathcal{W}_s, t\mathcal{W}_r)$ by the incompatibility of A and B. It follows that $\mathcal{W}_r \subset \Sigma$ is not possible, justifying $\mathcal{W}_r \subset s\Sigma$.

Let Λ be the sector of Σ satisfying $\mathcal{W}_r \subset s\Lambda$. It follows that $E_{J,I} \subset \Lambda$ and $E_{\tau(J),\tau(I)} \subset s\Lambda$. Consequently $E_{\tau(J),\tau(I)} = sE_{J,I}$. We also have $E_{\tau(I),\tau(J)} = E_{I,J}$ as in Case 1. Note that $E_{I,J}$ and $E_{J,I}$ are in distinct half-spaces for s. Now we can prove $d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) < d(E_{J,I}, E_{I,J})$ in the same way as in Case 1.

References

[1] Noel Brady, Jonathan P McCammond, Bernhard Mühlherr, and Walter D Neumann. Rigidity of coxeter groups and artin groups. *Geometriae Dedicata*, 94(1):91–109, 2002.

[2] Pierre-Emmanuel Caprace and Bernhard Mühlherr. Reflection rigidity of 2-spherical coxeter groups. *Proceedings of the London Mathematical Society*, 94(2):520–542, 2007.

[3] Pierre-Emmanuel Caprace and Piotr Przytycki. Twist-rigid coxeter groups. *Geometry & Topology*, 14(4):2243–2275, 2010.

[4] Ruth Charney and Michael Davis. When is a Coxeter system determined by its Coxeter group? *J. London Math. Soc. (2)*, 61(2):441–461, 2000.

[5] Jean-Yves Hée. Le cône imaginaire dune base de racines sur R. *Thèse d'état, Université d'Orsay*, 1990.

[6] R. B. Howlett, P. J. Rowley, and D. E. Taylor. On outer automorphism groups of Coxeter groups. *Manuscripta Math.*, 93(4):499–513, 1997.

[7] RB Howlett and B Mühlherr. Isomorphisms of coxeter groups which do not preserve reflections. *preprint*, 2004.

[8] Timothée Marquis and Bernhard Mühlherr. Angle-deformations in Coxeter groups. *Algebr. Geom. Topol.*, 8(4):2175–2208, 2008.

[9] Bernhard Mühlherr. The isomorphism problem for Coxeter groups. In *The Coxeter legacy*, pages 1–15. Amer. Math. Soc., Providence, RI, 2006.

[10] Bernhard Mühlherr and Richard Weidmann. Rigidity of skew-angled coxeter groups. *Adv. Geom*, 2(4):391–415, 2002.

[11] David G Radcliffe. Rigidity of right-angled coxeter groups. *arXiv preprint math/9901049*, 1999.

[12] John G. Ratcliffe and Steven T. Tschantz. Chordal Coxeter groups. *Geom. Dedicata*, 136:57–77, 2008.

[13] Mark Ronan. *Lectures on Buildings: Updated and Revised*. University of Chicago Press, 2009.

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