3–Manifolds, problem list 2

Problem 1. Show that for a closed 3–manifold M there is a bound on the size of a system $\mathcal{S}$ of disjoint spheres in M such that no component of $M - \mathcal{S}$ is B^3 or $S^2 \times I$.

Problem 2. Show that any I–bundle E with base a closed surface and ∂E a union of spheres is either $I \times S^2$ or its quotient by $\mathbf{Z}_2$ acting by reflection on I and the antipodal map on S^2 (the so called *twisted* bundle over $\mathbf{RP}^2$).

Problem 3. Let S^1 be a circle embedded in a surface Σ. Prove that if S^1 does not a bound a disc in Σ, then it is not contractible in Σ.

Problem 4. Compute the homology and cohomology groups of $\mathbf{RP}^3$ with coefficients $R = \mathbf{Z}, \mathbf{Z}_2, \mathbf{Q}$.

Note: if one wanted to compute the simplicial homology this would require choosing a triangulation of $\mathbf{RP}^3$, the simplest of which has 2 tetrahedra. But to compute the cellular homology (review the definition!) one is allowed to use any CW decomposition of $\mathbf{RP}^3$ into cells, and it is best to use the decomposition into one ball, one disc, one edge and one vertex.

Definition. Let M be a triangulated space and $A \subset M$ a subcomplex. *Simplicial relative homology* $H_*(M, A; R)$ is the homology obtained from the chain complex $C_*(M, A; R)$ with $C_i(M, A; R) = C_i(M; R)/C_i(A; R)$.

It turns out that $H_*(M, A; R) = H_*(M/A; R)$, the (cellular) homology of the quotient space. Moreover, there is a *long exact sequence* (meaning the image of each map is the kernel of the next map)

$$\ldots \to H_i(A; R) \to H_i(M; R) \to H_i(M, A; R) \to H_{i-1}(A; R) \to \ldots$$

Furthemore, *Poincaré duality* states that, if M is a compact oriented n–manifold, then $H_i(M, \partial M; R) \simeq H^{n-i}(M; R)$.

Problem 5. Let M be a compact oriented 3–manifold. Prove that

(i) if $H_1(M, \mathbf{Z})$ is finite, then all components of ∂M are spheres;

(ii) if $H_1(M, \mathbf{Z}) = \mathbf{Z}$, then one component of ∂M might be a torus and the remaining ones are spheres.

Hint: you can use the fact that $H^1(X; R) = \mathrm{Hom}(H_1(X; \mathbf{Z}), R)$.

Problem 6. Prove that a closed simply-connected 3–manifold M is homotopy equivalent to S^3. Hints:

(i) Prove that M is oriented.

(ii) Using Poincaré duality, compute $H_k(M; \mathbf{Z})$ (and $H_k(S^3; \mathbf{Z})$).

(iii) By Poincaré duality, $H_3(M; \mathbf{Z})$ equals $\mathbf{Z}$. If $\mathcal{T}$ is a triangulation of M, then show this using simplicial homology and characterise the generator of $H_3(M; \mathbf{Z})$ as the union of the tetrahedra of $\mathcal{T}$ (appropriately oriented).

(iv) By Whitehead and Hurewicz theorems, a sufficient condition for a map between manifolds (or CW complexes) to be a homotopy equivalence is to induce isomorphisms on π_1 and all H_k. Find such a map $M \to S^3$.