3–Manifolds

Midterm 2 preparation problems

Problem 1. What is the Euler number e of S^3 with the Hopf fibration? Hint: introduce an artificial singular point x on the base S^2 with $q(x) = 1$. Then $e = p(x)$.

Problem 2. Let M be the Poincaré sphere. Prove that M has Seifert fibering with base orbifold (S^2, X), where X consists of three points with multiplicities $2, 3, 5$. Hint: $SO(3)$ has a natural S^1–bundle structure coming from the identification with the unit tangent bundle of S^2.

Problem 3. Construct a spherical manifold with Seifert fibering whose base orbifold is (S^2, X), where X consists of three points with multiplicities $2, 3, 4$.

Problem 4. Let M be the space obtained by gluing the opposite sides of a cube via angle $\frac{\pi}{2}$ rotations. Prove that M is a manifold and find a model geometry carried by M.

Problem 5. Let M be the space obtained by gluing the opposite sides of a regular icosahedron via angle $\frac{\pi}{3}$ rotations. Is M a manifold?

Problem 6. Let P be a regular dodecahedron in $\mathbf{H}^3$ with dihedral angle $\frac{\pi}{2}$ (justify that P exists). Construct a closed hyperbolic manifold by gluing several copies of P.

Problem 7. Let M be a manifold obtained from gluing $T^2 \times [0, 1]$ by a map $(x, 0) \sim (Ax, 1)$, where $x \in T^2 = \mathbf{R}^2/\mathbf{Z}^2$ and

(i)
$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

(ii)
$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

(iii)
$$A = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$$

Find a model geometry carried by M.

Problem 8. Consider the unit sphere S^3 with the group structure coming from quaternions. Prove that the homomorphism $S^3 \times S^3 \to SO(4)$ assigning to a pair $g, h \in S^3$ the isometry $x \to g^{-1}xh$ is onto and 2 to 1.

Problem 9. Let $(G, \mathbf{H}^n)$ be the hyperbolic geometry. Suppose that $\Gamma \subset G$ acts properly and cocompactly on $\mathbf{H}^n$. Prove that Γ does not contain a nontrivial abelian normal subgroup. You are allowed to use

- the theorem that a finite subgroup of G fixes a point of $\mathbf{H}^n$,
- Selberg Lemma to claim that Γ has a torsion-free finite index subgroup.

Problem 10. Let $(G, \mathbf{H}^n)$ be the hyperbolic geometry. Suppose that $\Gamma \subset G$ acts properly on $\mathbf{H}^n$. Prove that Γ does not contain two loxodromic elements with exactly one common fixed point in $\partial\mathbf{H}^n$.