3–Manifolds

Midterm 1 preparation problems

Unless stated otherwise, all manifolds are connected, compact, and oriented.

Problem 1. Let V be the *genus g handlebody*: the 3–manifold bounded by the genus g oriented surface embedded in a standard way in $\mathbf{R}^3$. Let M be the manifold obtained by taking two copies of V and identifying their boundaries via the identity map. Find a decomposition of M into a connected sum of prime manifolds.

Problem 2. Let M be an irreducible 3–manifold and let $\Sigma \subset M$ be an incompressible surface. Prove that for each embedded disc $h\colon D \to M$ satisfying $h(D) \cap \Sigma = h(\partial D)$, there is a homotopy $H\colon D \times I \to M$ such that

- $H(\cdot, 0) = h$,
- $H(x, 1) \in \Sigma$ for each $x \in D$,
- $H(x, t) = H(x, t')$ for each $x \in \partial D, t, t' \in I$, and
- $H(x, t) \notin \Sigma$ for $t \neq 1$ and $x \notin \partial D$.

Problem 3. Prove that a 3–manifold M is contractible if and only if it is simply-connected and $\partial M = S^2$.

Problem 4. Let M be a 3–dimensional compact submanifold of $\mathbf{R}^3$. Show that if $H_1(M, \mathbf{Z}) = 0$, then $\pi_1(M) = 0$.

Problem 5. Let M be a possibly non-compact simply-connected 3–manifold. Show that each circle in ∂M separates ∂M (and consequently compact components of ∂M are spheres).

Problem 6. Prove that a closed 3–manifold has free fundamental group if and only if it is the connected sum of some copies of $S^2 \times S^1$ and manifolds homotopy equivalent to S^3 (not assuming the Poincaré Conjecture).

Problem 7. Find an example of a non-orientable surface Σ in an oriented 3–manifold M, such that Σ has no compressing discs but $\pi_1(\Sigma) \to \pi_1(M)$ is not injective.

Problem 8. Consider the orbifold $\theta = (S^2, X)$, where X consists of three points with multiplicities $2, 3, 3$. Compute $\chi(\theta)$. Prove that $\pi_1(\theta) = A_4$ (even permutations of the 4–element set). Hint: find an action of A_4 on S^2 with quotient θ.

Problem 9. Consider the orbifold $\theta = (S^2, X)$, where X consists of three points with multiplicities 3. Compute $\chi(\theta)$. Prove that $\pi_1(\theta)$ has a finite index subgroup isomorphic with $\mathbf{Z}^2$. Hint: find an action of a group on $\mathbf{R}^2$ with quotient θ.

Problem 10. Consider the orbifold $\theta = (S^2, X)$, where X is one point, or two points with multiplicities $p \neq q$. Prove that θ is not a quotient of an action of a group on a surface.