3–Manifolds

Final exam material

(1) Definitions of a smooth manifold, diffeomorphism, immersion, triangulation, orientation, embedding. Classification statements in dimension 1 and 2.

(2) Definition of an isotopy and the statement about its extension. Statement of Cerf–Palais theorem. Definition of a connected sum, correspondence with cutting along spheres in dimension $\leqslant 3$. Statement of Alexander's theorem. Definition of prime and irreducible 3–manifolds; proof of proposition relating these two notions. Definition and lemmas on transversality. Definition of a properly embedded surface.

(3) Prime decomposition theorem with proof. Definition of a fiber bundle.

(4) Definition of an incompressible surface. Loop Theorem with corollaries and proof. Statement of Sphere Theorem. Discussion of possible $\widetilde{M}$ for M irreducible. Statement of Poincaré Conjecture.

(5) Definition of ∂–parallel surfaces, and atoroidal 3–manifolds. Definition of a Seifert manifold, and multiplicity of fibers. Statement of Jaco–Shalen–Johannson theorem. Definition of an orbifold and its Euler characteristic. Construction of Seifert manifolds and definition of Euler number e.

(6) Definition of a Lie Group and model geometry. Examples and classification in dimension 2. Complete proof that the Poincaré sphere has trivial H_1.

(7) Examples of 3-dimensional Euclidean manifolds. Definition of a crystalographic group and the statement of Bieberbach theorem. Construction of Seifert–Weber dodecahedral space and figure 8 knot complement.

(8) Statement and proof of the proposition about discrete and cocompact subgroups of $\mathrm{Isom}(\mathbf{H}^2 \times \mathbf{R})$. Proof of its corollary that manifolds carrying $\mathbf{H}^2 \times \mathbf{R}$ geometry are Seifert. Definition of $\widetilde{\mathbf{PSL}}(2, \mathbf{R})$ geometry. Definition of a connection and associated metric, examples.

(9) Definition of Nil geometry. Definition of the Heisenberg group, and its description as the fundamental group of a torus bundle. Definition of Sol geometry. Proof of the proposition and corollary that manifolds carrying Sol geometry are particular torus bundles.

(10) Flowchart to distinguish between fundamental groups of closed 3-manifolds carrying different geometries. Table of $\chi(\theta)$ and e for closed Seifert manifolds carrying different geometries. Statement of Thurston's Geometrization Conjecture.

(11) Definition of a Kleinian group Γ and its limit set $\Lambda(\Gamma)$. Proof of the theorem that $\Lambda(\Gamma) = \Lambda(a)$. Definition of domain of discontinuity, examples.

(12) Definition of injectivity radius, thin and thick parts, tubes and cusps. Full proof of the structure of the thin part, including Kazhdan–Margulis and Zassenhaus theorems with proofs. Statement of Mostow rigidity theorem.