3–Manifolds, problem list 8

Definition. *Hyperbolic geometry* $(G, \mathbf{H}^n)$ in dimension n is the pair $\mathbf{H}^n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | \; x_n > 0\}$ and G the group of *Möbius transformations* of $\mathbf{H}^n$, that is the group generated by reflections in hyperspaces and inversions in hyperspheres orthogonal to the hyperplane $x_n = 0$. The action of G extends to $\partial\mathbf{H}^n$ defined as the hyperplane $x_n = 0$ compactified by a point *at* ∞.

Problem 1. Show that for any pair of ordered triples of points of $\partial\mathbf{H}^n$, there is an element of G mapping the first triple to the second one.

Problem 2. Prove that each element of G is exactly of one of the following types:

1. *elliptic*: preserving a point of $\mathbf{H}^n$,
2. *parabolic*: conjugate to $(x_1, \ldots, x_n) \to (f(x_1, \ldots, x_{n-1}), x_n)$, where $f \in \mathrm{Isom}(\mathbf{R}^{n-1})$ with no fixed points,
3. *loxodromic (or hyperbolic)*: conjugate to $(x_1, \ldots, x_n) \to \lambda(f(x_1, \ldots, x_{n-1}), x_n)$, where $f \in O(n-1)$, the orthogonal group of $\mathbf{R}^{n-1} = (x_1, \ldots, x_{n-1})$, and $\lambda \neq 1$.

Hint: Brouwer fixed point theorem for $\mathbf{H}^n \cup \partial\mathbf{H}^n$.

Problem 3. Suppose that $\Gamma \subset G$ acts properly and cocompactly on $\mathbf{H}^n$. Prove that Γ does not contain parabolic elements.

Hint: Let $|\cdot, \cdot|$ be a G–invariant metric on $\mathbf{H}^n$. If $g \in \Gamma$ is parabolic, there are $x_n \in \mathbf{H}^n$ with $|g(x_n), x_n| < \frac{1}{n}$. Find $x_0 \in \mathbf{H}^n$ and $h_n \in G$ with $|x_n, h_n(x_0)|$ converging to 0. Prove that infinitely many of $h_n^{-1} g h_n$ coincide and deduce that g fixes a point of $\mathbf{H}^n$.

Problem 4. Let $G_+ \subset G$ be the index 2 subgroup of orientation preserving Möbius transformations of $\mathbf{H}^n$ and let $n = 2$. Prove that for each abelian subgroup $A \subset G_+$ all elements in $A - \mathrm{Id}$ are of the same type (elliptic, parabolic or loxodromic) and have the same fixed-point set in $\mathbf{H}^2 \cup \partial\mathbf{H}^2$.