3–Manifolds, problem list 7

Problem 1. Find a formula for an element of G in Nil geometry descending to a rotation of the x, y plane around the origin.

Problem 2. Prove that the Heisenberg group is nilpotent.

Hint: Let $f, g \in G$ in Nil geometry be elements descending to translations of the x, y plane. What is $[f, g]$?

Problem 3. Prove that each compact manifold carrying Nil geometry is Seifert with $\chi(\theta) = 0$ and $e \neq 0$.

Hint: mimic the proof for $\mathbf{H}^2 \times \mathbf{R}$.

Problem 4. Let Γ be the fundamental group of the 3–manifold obtained from $T^2 \times I$ by gluing its boundary components via $\phi \in SL(2, \mathbf{Z})$ that has two distinct real eigenvalues. Show that Γ is not nilpotent.

Problem 5. Show that each nontrivial solvable group has a nontrivial normal subgroup that is abelian.