3–Manifolds, problem list 5

Problem 1. Let M be one of the closed Seifert manifolds constructed in class. Prove that there exists a surface in M transverse to all the fibers if and only if $e = 0$.

Hint: for the "if" direction, let Q be the least common multiple of all $q(x)$. Start with Q parallel copies of Σ' in M'. Cut them and reglue along arcs joining different boundary components to be able to cap them off using $\frac{Q}{q(x)}$ discs transverse to singular fibers over x.

Problem 2. Let M be a manifold that is a quotient G/G_p of a Lie group G by its compact subgroup. Let Γ be a discrete subgroup of G such that its natural action on M is free. Prove that $M \to M/\Gamma$ is a covering map.

Hint: prove that the action of Γ on M is *proper*, that is for each ball $B \subset M$ the set of $\gamma \in \Gamma$ with $B \cap \gamma B \neq \emptyset$ is finite.

Problem 3. Prove that $SO(3)$ is homeomorphic with $\mathbf{RP}^3$. Hints:

(i) View $\mathbf{R}^4$ as *quaternions*: name orthonormal basis vectors by $1, i, j, k$ and define multiplication by $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji, jk = i = -kj, ki = j = -ik$ (extend linearly and check associativity). Show that $|pq| = |p||q|$ and consequently multiplication by a unit quaternion from left (or right) is an isometry of $\mathbf{R}^4$. In particular $S^3 \subset \mathbf{R}^4$ is a group.

(ii) Let $S^2 \subset S^3$ be the intersection of S^3 with the subspace spanned by i, j, k. Prove that the action of S^3 on itself by conjugation preserves S^2, and in fact is onto $SO(3)$ with kernel $\{1, -1\}$.

Problem 4. Prove that the Poincaré sphere M has the same homology groups as S^3.

Hint: to exclude $H_1(M, \mathbf{Z}) = \mathbf{Z}_2$, inscribe an icosahedron into S^2 above so that i and j act as its orientation-preserving isometries. Compute the commutator $[i, j] = -1$.

Problem 5. Let P be the regular dodecahedron. Let M be the space obtained from gluing opposite faces of P via an angle $\frac{\pi}{5}$ rotation. Prove that M is spherical. Hint: embed P in S^3 as a regular dodecahedron with dihedral angle $\frac{2\pi}{3}$.

Note: In fact M is the Poincaré sphere, but this is not easy to prove.