3–Manifolds, problem list 4

Problem 1. Let Σ be a closed non-orientable surface Σ in an oriented 3–manifold M. (Then a neighbourhood N of Σ is the *twisted I–bundle over Σ*.) Prove that $\pi_1(\Sigma) \to \pi_1(M)$ is injective if and only if ∂N is incompressible in M.

Problem 2. Let M be a 3–manifold and let $S^2 \subset M$ be contractible in M. Show that S^2 bounds a simply-connected submanifold N of M.

Hint: mimic the solution to Problem 4 from list 1.

Problem 3. Let M be a 3–manifold such that each map $S^2 \to M$ is contractible in M. Show that M is irreducible.

Hint: you are allowed to use the Poincaré Conjecture that each 3–manifold homotopy equivalent to S^3 is diffeomorphic to S^3. Apply it to $N \cup B^3$, where N is the submanifold from Problem 2.

Problem 4. Show that for a compact irreducible 3–manifold M there is a bound on the size of a system $\mathcal{T}$ of disjoint incompressible tori in M such that no component of $M - \mathcal{T}$ is of form $T \times I$.