3–manifolds, problem list 1

Unless stated otherwise, all manifolds are compact and connected.

Problem 1. Suppose that M_1, M_2 are oriented 3-manifolds and $M = M_1 \# M_2$. Prove that $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$ (free product). Hint: consider $M_i' = M_i - B^3$, and prove first $\pi_1(M) = \pi_1(M_1') * \pi_1(M_2')$.

Problem 2. The *three dimensional projective space* $\mathbf{RP}^3$ is the manifold obtained from the three sphere S^3 by quotienting by the antipodal map. Is $\mathbf{RP}^3$ orientable (find charts)?

Problem 3. Find an orientable manifold covered by $S^2 \times S^1$ that is not prime. (Hint: find appropriate action of $\mathbf{Z}_2$ on $S^2 \times S^1$.)

Problem 4. Prove that if a 3–manifold M' is irreducible, then any manifold M it covers is also irreducible. (Hint: for each sphere S^2 in M consider all its lifts to M', and find a lift that bounds a ball disjoint from the other lifts.)

Problem 5. Let $q \geq 2$ and $0 < p < q$ be relatively prime with q. Prove that the following definitions of the lens space $L_{p,q}$ are equivalent:

(i) 3–manifold obtained from the unit sphere S^3 in $\mathbf{C}^2$ by quotienting by $\mathbf{Z}_q$ generated by the rotation $(z_1, z_2) \to (e^{2\pi i \frac{1}{q}} z_1, e^{2\pi i \frac{p}{q}} z_2)$. (In particular $L_{1,2} = \mathbf{RP}^3$.)

(ii) On the boundary S^2 of the ball B^3 we identify the following points: each point x of the upper hemisphere we identify with the point of the lower hemisphere obtained from x by the reflection through the equator and rotation about the axis joining the poles by the angle $2\pi\frac{p}{q}$.

(iii) We glue two solid tori $S^1 \times D^2$ along their boundary by mapping ∂D^2 of one torus to the circle of slope $\frac{p}{q}$ of the other. (Why does the gluing not depend on where we map S^1 of the first one?)

Here the *slope* of a circle $C \subset S^1 \times \partial D^2$ is computed according to the following convention: The torus $S^1 \times \partial D^2$ is represented as glued from a square with S^1 the horizontal edge and ∂D^2 the vertical edge. Each circle C can be homotoped to be formed of parallel straight line segements and the tangent of their angle with respect to the horizontal direction is called the *slope* of C. For example the slope of S^1 is 0 and the slope of ∂D^2 is ∞.