Random groups of intermediate rank

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I provide additional information on a recent paper of Sylvain Barré and myself [1]. The goal is to construct a new family of (finitely presented countable) groups which are "almost" lattices in certain higher rank Lie groups over nonarchimedean local fields (for example $K = \mathbb{F}_p((x))$ and $\mathbf{G} = \mathrm{SL}_3$).

We introduce new models of random groups which are parallel to the classical models of Gromov [3, 4]. Let us first describe the latter briefly.

Start with the free group F_2 on 2 generators a, b (or more generally any group that has "many quotients") and let

$$W_n = \text{the set of cyclically reduced words of length } n \text{ in } a^{\pm 1}, b^{\pm 1}.$$

M. Gromov defines several models of random groups, for example:

The few-relators model. Let $\ell \geq 1$ be a fixed integer. Choose uniformly and independently at random ℓ words $w_1, \ldots, w_\ell$ in W_n (we omit the dependance of w_i in the particular event and n) and consider the group presentation

$$\langle F_2 \mid w_1, \ldots, w_\ell \rangle.$$

Say that a property is satisfied with *overwhelming probability* if the probability to satisfy it goes to 1 as n goes to infinity. Gromov [3] proves that for every $\ell \geq 1$, a random group in this model is hyperbolic with overwhelming probability (and more precisely, that it satisfies the $C'(\lambda)$ small cancellation condition for every $\lambda > 0$, as all relations have same length).

The density model. In this model, the number ℓ of randomly chosen relations (again, uniformly and independently) is allowed to vary with n. Typically, M. Gromov chooses $\ell_n = |W_n|^d$ words in W_n, where d is a (density) parameter, and constructs a random group G with presentation

$$\langle F_2 \mid w_1, \ldots, w_{\ell_n} \rangle.$$

A well-known result of [4] asserts that if $d > 1/2$, then G is trivial with overwhelming probability (trivial means $G = \{e\}$ or $G = \mathbb{Z}/2\mathbb{Z}$ here), while if $d < 1/2$ then G is infinite hyperbolic with overwhelming probability.

In the new models, we will rather start with a sequence of finitely presented group G_n, called the *initial data* for the model, and *remove* relations at random from the given presentation of G_n. The number of relations in the given presentation of G_n tends to infinity. Depending on how we remove these relations, we obtain analogs of the Gromov models above (see [1]).

At this level of generality, this is of little use but the point is to choose interesting sequences of finitely presented groups G_n as initial data, which typically arise as a decreasing chain of finite index subgroups of a well–chosen G_1.

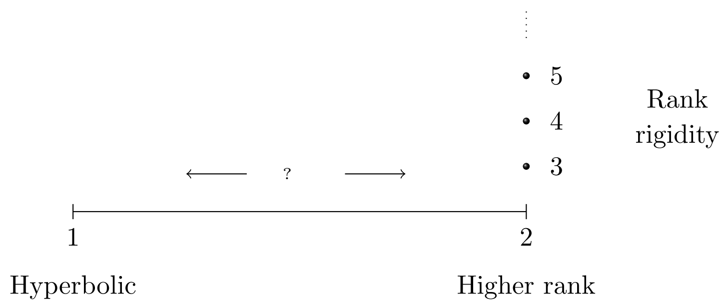
Thus, there are two important points with the new models:

(1) They are "mirror symmetric" to the Gromov models, in the sense that, rather than adding relations at random to groups which have "many quotients", we remove relations at random to groups which have (very few quotients but) "many extensions".
(2) They are *localized* in the space of finitely presented groups: rather than being generic among all finitely presented groups, we will restrict these constructions to a tiny portion of the space of group presentation and prove results there. The model can be seen as a *randomization* of the initial sequence of group presentations.

An example where (2) is not satisfied would be to let the sequence G_n densify in the space of finitely generated groups (endowed with the usual Chabauty topology, i.e. pointed Gromov–Hausdorff).

We are interested in drawing conclusions in "geometry of intermediate rank" (see [2]), whence our initial data are of appropriate geometric (and arithmetic) origin.

Let K be a nonarchimedean local field (e.g. $K = \mathbb{Q}_p$ or $K = \mathbb{F}_p((x))$). Let us choose as initial data for the model certain arithmetic chains $G_1 \geq G_2 \geq \ldots$ of finite index normal subgroups in a uniform lattice of $\mathrm{SL}_3(K)$. For instance, take the Lubotzky–Samuels–Vishne congruence subgroups, which one can have a hold on using strong approximation (see [5]). The Bruhat–Tits buildings X_K of $\mathrm{SL}_3(K)$ is of rank 2, and the randomization of G_n is obtained by removing at random (uniformly and independently) G_n-orbits of chambers in X_K. Equivalently, if the action of G_1 on X_K is free, we remove at random chambers of X_K/G_n. Taking the universal cover (say X) of this (random) space, we obtain a random group G which acts freely uniformly on the CAT(0) space X and which surjects to a uniform lattice in $\mathrm{SL}_3(K)$—namely, G_n (see [1] for details). It is convenient to introduce a notion of "building with chambers missing" to explain the properties of (X, G). As described above, we can then consider analogs of the few-relators model and the density model, called respectively the model with a few chambers missing and the lattice density model.



The idea of *rank interpolation* is quite self-explanatory. Our setting is that of CAT(0) space X (usually with a simplicial structure) endowed with a proper action of a countable group G with compact quotient. In this context, the word "rank" refers to everything related to *flatness* in X. We want to interpolate between "soft" hyperbolic spaces (which have rank 1) and rigid "higher rank" spaces, like Bruhat–Tits buildings or symmetric spaces of rank at least 2. Many concrete examples are given in [2], for example we find there spaces of so-called "rank $\frac{7}{4}$".

Here is a specific open problem: if X contains an isometric copy of $\mathbb{R}^2$, does G contains a copy of $\mathbb{Z}^2$? This question is a version of the flat closing conjecture, which has received a lot of attention over the years. The above-mentioned intermediate rank constructions provide many test spaces (X, G) for it. The randomized lattices too are of intermediate rank. In fact their rank is "very close to 2", in various senses that can be made precise. For example, as far as the above question is concerned, we prove that, in the lattice density model with initial data from [5], the group G contain a copy of $\mathbb{Z}^2$ whenever $d < 1/4$. Like in Gromov's model, there is a phase transition at $d = 1/2$, so that (with overwhelming probability) the group G splits off a free group $G = G_0 * F_k$ if $d > 1/2$, while it has Kazhdan's property T if $d < 1/2$ (if the order of the residue field is large enough). It would be interesting to determine whether the existence of $\mathbb{Z}^2$ still holds for $d > 1/4$, especially for $1/4 < d < 1/2$ (there are some choices of initial data from [5] for which it does). The existence of $\mathbb{R}^2$ in X holds with overwhelming probability whenever $d < 1/2$ in all cases. Several problems are left open. For example: what rigidity properties (Mostow, Margulis, OE,...) do these random groups have? Are they non linear? residually finite?...

Like in many models of random groups, the groups we obtain here have geometric dimension 2. The same construction can be performed in higher dimension but will not lead to new groups. Taking for instance the Ramanujan hypergraphs associated (by [5]) to chains of subgroups of $\mathrm{SL}_n(K)$ for $n \geq 3$, the above randomization does produce random hypergraphs with interesting expansion properties. Again, these random hypergraphs are localized in the space of hypergraphs, as opposed to the usual constructions of random graph, for example, used to show that "random graphs are expanders".

These constructions belong to the usual "structure and randomness" circle of idea, although it goes the opposite way: rather than uncovering the structure of some random/huge object (as in the Szemerédi lemma for example), we start with a structured set of data and add a random component to it.

References

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