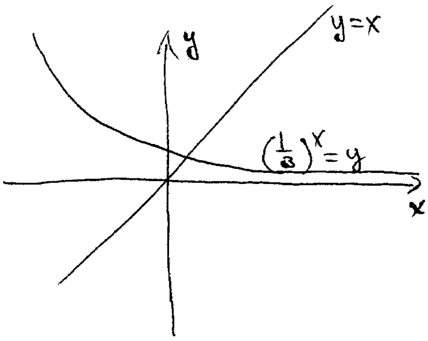
MAST 334 / MATH 354
Assignment #1, Solutions



Pr 2a $\quad x - 3^{-x} = 0 \quad$ or $\quad x = \left(\frac{1}{3}\right)^x$

$f(x) = x - \left(\frac{1}{3}\right)^x$

$f(0) = -1 < 0, \quad f(1) = 1 - \frac{1}{3} > 0$

Answer: for example $(0,1)$.

3b at the end

Pr 11 $\quad f(x) = (x-1)\ln x \quad , x_0 = 1$

$f'(x) = \ln x + \frac{x-1}{x} = \ln x + 1 - \frac{1}{x} \qquad f'(1) = 0$

$f''(x) = \frac{1}{x} + \frac{1}{x^2} \qquad f''(1) = 2$

$f'''(x) = -\frac{1}{x^2} - \frac{2}{x^3} \qquad f'''(1) = -3$

$f^{IV}(x) = \frac{2}{x^3} + \frac{6}{x^4}$

$P_3(x) = (x-1)^2 - \frac{1}{2}\cdot(x-1)^3 \; ; \; R_3(x) = \frac{1}{4!}\left(\frac{2}{\xi(x)^3} + \frac{6}{\xi(x)^4}\right)(x-1)^4$

$P_3(0.5) = (0.5)^2 + \frac{1}{2}(0.5)^3 = 0.3125$

$|R_3(\underset{\uparrow 0.5}{x})| \le \frac{1}{4!}\left(\frac{2}{(.5)^3} + \frac{6}{(.5)^4}\right)(0.5)^4 = \frac{1}{24}(2\cdot 0.5 + 6) = \frac{7}{24} = 0.2917$

Pr 11 (continued)

actual error : $P_3(0.5) - 0.5 \ln 0.5 = 0.3125 - 0.34657$
≈ 0.034073

b). $|f(x) - P_3(x)| \le |R_3(x)| \quad x \in [0.5, 1.5]$
The estimate is the same as in previous part.

c) $$\int_{0.5}^{1.5} f(x)\,dx \approx \int_{0.5}^{1.5} (x-1)^2 - \tfrac{1}{2}(x-1)^3\,dx =$$
$$= \tfrac{1}{3}(x-1)^3 - \tfrac{1}{8}(x-1)^4 \Big]_{0.5}^{1.5}$$
$$= \tfrac{1}{3}(0.5)^3 - \tfrac{1}{8}(0.5)^4 - \left(\tfrac{1}{3}(-0.5)^3 - \tfrac{1}{8}(-0.5)^4\right)$$
$$= \tfrac{2}{3}\cdot(0.5)^3 = \tfrac{1}{12} \approx 0.0833$$

actual integral : $$\int_{0.5}^{1.5} (x-1)\ln x\,dx = \int_{0.5}^{1.5} x\ln x - \ln x\,dx =$$
$$= \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 - x\ln x + x \Big]_{0.5}^{1.5} = 0.08802$$

Actual error : $0.0833 - 0.08802 \approx 0.0046873$

d) Estimation of the error:
$$\int_{0.5}^{1.5} |R_3(x)| \le \tfrac{1}{24}\int_{0.5}^{1.5} \left(\frac{2}{(0.5)^3} + \frac{6}{(0.5)^4}\right)(x-1)^4\,dx$$
$$= \tfrac{1}{24}(16 + 6\cdot 16)\ \tfrac{1}{5}(x-1)^5 \Big]_{0.5}^{1.5}$$
$$= \tfrac{1}{24}(7\cdot 16)\cdot\tfrac{1}{5}\left((+\tfrac{1}{2})^5 - (-\tfrac{1}{2})^5\right)$$
$$= \tfrac{1}{24}\,7\cdot 16\cdot\tfrac{1}{5}\cdot\tfrac{2}{32} \approx \tfrac{7}{120} \approx 0.0583333\ldots$$

Pr. 21 $\quad f(x) = \cos x \qquad x_0 = 0$

$f(0) = 1$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'(0) = 0$

$f''(0) = -1$

$f'''(x) = \sin x$

$f'''(0) = 0$

$f^{IV}(x) = \cos x$

$$P_3(x) = P_2(x) = 1 - \frac{1}{2}x^2$$

$$|f(x) - P_2(x)| \leq |R_3(x)| = \left| \frac{1}{4!} \cos(\xi(x)) \cdot x^4 \right|$$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$

Thus, $\quad |R_3(x)| \leq \frac{1}{24} \cdot \frac{1}{2^4} = \frac{1}{24 \cdot 16} \approx 0.002604$

Pr. 5 $\quad$ a) exact : 133.921

3-digit rounding : $133.921 = 134$

error : 0.079

relative error : $\frac{|133.921 - 134|}{133.921} \approx 0{,}0005899$

d) $(121-119) - 0.327 = 2 - 0.327 = 1.673$ exact

$= 1.67$ 3 digit rounding

error $|1.673 - 1{,}67| = 0{,}003$

relative error $\quad \frac{0.003}{1.673} = 0.0017931$

h) $\dfrac{\pi - \frac{22}{7}}{1/17}$ exact $\pi = 3.141592654$

$\frac{22}{7} = 3.142857143$

$\pi - \frac{22}{7} = -0.001264489$

$1/17 = 0.058823529$

result $= -0.021496313$

3-digit rounding $\pi = 3.14$ $\quad \pi - \frac{22}{7} = 0$

$\frac{22}{7} = 3.14$

result $= 0$

error $|0 + 0.021496313| = 0.021496313$

relative error 100%.

Problem 36:

$$f(x) = (x-1)\tan x + x \sin \pi x \qquad x \in [0,1]$$

$f(0) = -1 \cdot 0 + 0 \cdot 0 = 0$

$f(1) = 0 \cdot \tan 1 + 1 \sin \pi = 0$

By Rolle's theorem there exists an $x_0 \in (0,1)$ such that $f'(x_0) = 0$.