# REMARK ON STABILITY OF TRAVELING WAVES FOR NONLOCAL FISHER-KPP EQUATIONS

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**Abstract.** This paper is concerned with a class of nonlocal Fisher-KPP type reaction-diffusion equations in $n$-dimensional space with time-delay. It is proved that, all noncritical planar wavefronts are exponentially stable in the form of $t^{-\frac{n}{2}}e^{-\nu_\tau t}$ for some constant $\nu_\tau = \nu(\tau) > 0$, where $\tau \geq 0$ is the time-delay, while the critical planar wavefronts are algebraically stable in the form of $t^{-\frac{n}{2}}$. These convergent rates are optimal in the sense with $L^1$-initial perturbation. The adopted approach is the weighted energy method combining Fourier transform. It is also realized that, the effect of time-delay essentially causes the decay rate of the solution slowly down. These results significantly generalize and develop the existing study [37] for 1-D time-delayed Fisher-KPP type reaction-diffusion equations. When the time-delay $\tau$ vanishes, we automatically obtain the exponential stability for the noncritical planar traveling waves and the algebraic stability for the critical planar traveling waves to the regular Fisher-KPP equations.

**Key words.** Nonlocal reaction-diffusion equations, time delays, traveling waves, global stability, the Fisher-KPP equation, $L^1$-weighted energy, Green functions.

## 1. Introduction and Main Results

Following the recent study [37] on the stability of traveling waves to 1-D nonlocal time-delayed reaction-diffusion equations, in this paper, we study a class of $n$-D nonlocal Fisher-KPP reaction-diffusion equations ([4, 11, 25, 37])

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u + d(u) = \displaystyle\int_{\mathbf{R}^n} f_\alpha(y)b(u(t-\tau, x-y))dy, \\ u|_{t=s} = u_0(s,x), \ x \in \mathbf{R}^n, s \in [-\tau, 0] \end{cases}$$

for $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$ and $t \geq 0$. Here, $\Delta u = \displaystyle\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$, $D > 0$ is the diffusion coefficient, $\tau \geq 0$ is the time-delay, $f_\alpha(y)$, with $\alpha > 0$, is the heat kernel in the form of

$$(2) \qquad f_\alpha(y) = \frac{1}{(4\pi\alpha)^{\frac{n}{2}}} e^{\frac{-|y|^2}{4\alpha}} \quad \text{with} \quad \int_{\mathbf{R}^n} f_\alpha(y)dy = 1,$$

$d(u)$ and $b(u)$ both are nonlinear functions satisfying

(H$_1$) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = 0$, $d(u_+) = b(u_+)$, and $d(u), b(u) \in C^2[0, u_+]$;

(H$_2$) $b'(0) > d'(0) \geq 0$ and $0 \leq b'(u_+) < d'(u_+)$;

(H$_3$) For $0 \leq u \leq u_+$, $d'(u) \geq 0$, $b'(u) \geq 0$, $d''(u) \geq 0$, $b''(u) \leq 0$.

The model of (1) describes the wave propagations in fluid dynamics, and in physical, chemical and biological dynamics, initially given by R.A. Fisher [10], and A. Kolmogoroff, I. Petrovsky and N. Piscounoff [22]. The study on such a wave propagation phenomenon can be also found in [1, 31] for the fluid dynamical experiments on Taylor-Couette flow, in [7] for Rayleigh-Benard flow, in [44, 52] for the

chemical wave experiments, and in [3] for population dynamics, combustion, and biological invasions.

In the equation (1), if we take $\tau = 0$ and $\alpha \to 0^+$, and use the property of heat kernel $f_\alpha(y)$:

$$(3) \qquad b(u(t, x)) = \lim_{\alpha \to 0^+} \int_{\mathbf{R}^n} f_\alpha(y) b(u(t, x - y)) dy,$$

we derive the following regular Fisher-KPP reaction-diffusion equation [3, 10, 9, 15, 53, 55]

$$(4) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u = h(u), \\ u|_{t=0} = u_0(x), \ \ x \in \mathbf{R}^n, \end{cases}$$

with $h(u) = b(u) - d(u)$. Particularly, taking $d(u) = u^2$ and $b(u) = u$, then we reduce (4) to the following classical Fisher-KPP equation [3, 8, 10, 12, 21, 22, 41, 43]

$$(5) \qquad \frac{\partial u}{\partial t} - D\Delta u = u(1 - u), \ \ t > 0, x \in \mathbf{R}^n.$$

Clearly, from (H$_1$), both $u_- = 0$ and $u_+ > 0$ are constant equilibria of the equation (1); and from (H$_2$), $u_- = 0$ is unstable and $u_+$ is stable for the spatially homogeneous equation associated with (1); and from (H$_3$), both $b(u)$ and $d(u)$ are increasing, and $b(u)$ is concave downward and $d(u)$ is concave upward. These characters let the equations (1) and (4) capture the most basic features of the classical Fisher-KPP equation (5), so we call the equations (1) and (4) as the nonlocal/local Fisher-KPP type reaction-diffusion equations. Except the standard example with $b(u) = u$ and $d(u) = u^2$ for the classical Fisher-KPP equation (5), equation (1) includes the other two important examples. One is the Nicholson's blowflies equation [27, 28, 30, 35, 36, 37, 38, 39, 47, 48]

$$\frac{\partial u}{\partial t} - D\Delta u + \delta u(t, x) = \varepsilon p \int_{\mathbf{R}^n} f_\alpha(y) u(t - \tau, x - y) e^{au(t-\tau, x-y)} dy,$$

with

$$b(u) = \varepsilon p u e^{-au} \ \text{ and } \ d(u) = \delta u, \quad \varepsilon > 0, \ p > 0, \ a > 0, \ \delta > 0.$$

Obviously, these specified functions $b(u)$ and $d(u)$ satisfy (H$_1$)-(H$_3$) with $u_- = 0$ and $u_+ = \frac{1}{a} \ln \frac{\varepsilon p}{\delta}$ for $1 < \frac{\varepsilon p}{\delta} \le e$. The other typical example is the age-structured population model [2, 13, 14, 26, 37, 40]

$$\frac{\partial u}{\partial t} - D\Delta u + \delta u^2(t, x) = p e^{-\gamma\tau} \int_{\mathbf{R}^n} f_\alpha(y) u(t - \tau, x - y) dy,$$

with

$$d(u) = \delta u^2 \ \text{ and } \ b(u) = p e^{-\gamma\tau} u, \quad \delta > 0, \ p > 0, \ \gamma > 0,$$

which also satisfy (H$_1$)-(H$_3$) automatically with $u_- = 0$ and $u_+ = \frac{p}{\delta} e^{-\gamma\tau}$.

A *planar traveling wavefront* to the equation (1) is a special solution in the form of $u(t, x) = \phi(x \cdot \mathbf{e} + ct)$ with $\phi(\pm\infty) = u_\pm$, where $c$ is the wave speed, $\mathbf{e}$ is a unit vector of the basis of $\mathbf{R}^n$. Without loss of generality, we can always assume $\mathbf{e} = \mathbf{e}_1 = (1, 0, \cdots, 0)$ by rotating the coordinates. Thus, we have the planar traveling wavefront in the form $\phi(x \cdot \mathbf{e}_1 + ct) = \phi(x_1 + ct)$, which satisfies, for $\tau \ge 0$,

$$(6) \qquad \begin{cases} c\phi' - D\phi'' + d(\phi) = \displaystyle\int_{\mathbf{R}^n} f_\alpha(y) b(\phi(\xi_1 - y_1 - c\tau)) dy, \\ \phi(\pm\infty) = u_\pm, \end{cases}$$

where $' = \frac{d}{d\xi_1}$ and $\xi_1 = x_1 + ct$. Let

$$f_{\alpha i}(y_i) := \frac{1}{(4\pi\alpha)^{1/2}} e^{-\frac{y_i^2}{4\alpha}}. \tag{7}$$

Then

$$f_\alpha(y) := \prod_{i=1}^{n} f_{\alpha i}(y_i), \quad \text{and} \quad \int_{\mathbf{R}} f_{\alpha i}(y_i) dy_i = 1, \quad i = 1, 2, \cdots, n, \tag{8}$$

and (6) is reduced to, for $\tau \geq 0$,

$$\begin{cases} c\phi' - D\phi'' + d(\phi) = \int_{\mathbf{R}} f_{\alpha 1}(y_1) b(\phi(\xi_1 - y_1 - c\tau)) dy_1, \\ \phi(\pm\infty) = u_\pm. \end{cases} \tag{9}$$

The main purpose of this paper is to study the global asymptotic stability of planar traveling wavefronts of (1), including the case of the *critical wave* $\phi(x_1 + c_* t)$. Here the number $c_*$ is called the *critical speed* (or the *minimum speed*) in the sense that a traveling wave $\phi(x_1 + ct)$ exists if $c \geq c_*$, while no traveling wave $\phi(x_1 + ct)$ exists if $c < c_*$.

The study on the stability of traveling waves for reaction-diffusion equations has been a popular research area. There are many significant contributions on this topic, see, e.g., [5, 6, 9, 12, 17, 19, 21, 24, 32, 41, 43, 50], the monograph [53], the survey paper [55], and the references therein. In particular, the stability of the critical traveling waves is most interesting in fluid dynamics and biological invasions, but also very challenging. For the regular 1-D Fisher-KPP equation (4), particularly, the classical Fisher-KPP equation (5), Sattinger [43] first proved that all non-critical waves are exponentially stable by the spectral analysis method. Later on, Uchiyama [51] showed the local stability for the traveling waves including the critical waves by the maximum principle method, but no convergence rate for the critical waves case was related. In [5], Bramson derived the sufficient and necessary condition for the stability of noncritical and critical waves (no convergence rates), which was also obtained by Lau [24] later in a different way. Moet [41] showed that the critical waves are algebraically stable in the form of $O(t^{-1/2})$ by the Green function method. Kirchgässner [21] also obtained the stability for the critical waves in the form $O(t^{-1/4})$ by the spectral method, which was further improved to be $O(t^{-3/2})$ by Gallay [12] by using the renormalization group method, of course, the corresponding weight function needed to be stronger. For the multi-dimensional case, the stability of planar faster traveling waves with $c > c_*$ was obtained by Mallordy and Roquejoffre in [33], see also [18] for the stability on the manifolds but without convergence rates. Recently, Mei, Ou and Zhao [37] obtained the exponential stability for the non-critical traveling waves and the algebraic stability for the critical waves to the 1-D nonlocal time-delayed reaction-diffusion waves by the $L^1$ weighted energy method together with the Green function method, which also includes the above mentioned stability results for Fisher-KPP equations by taking the time-delay $\tau = 0$. However, the proof for deriving the rate $O(t^{-1/2})$ for the case of critical waves in Lemma 3.7 in [37] is not rigorous (for details, we refer to Remark 2.4 below).

There are three issues considered in this paper. The first is to fix the gap in [37] for obtaining the algebraic convergence rate $O(t^{-1/2})$ to the critical traveling waves. As we know, in [37] they converted the working equation to the integral form with the regular Green function (the heat kernel without time-delay), then used the iteration procedure to derive the algebraic convergence rate in the case of

critical waves: $C^k(1+t)^{-1/2}$ at the $k$th iteration. So, the constant coefficient $C^k$ is increasing and unbounded as $k \to \infty$. In order to fix such a gap, here, we technically derive the equivalent integral equation with the time-delayed Green function, and show the optimal decay rates of the solutions without iteration. The second is to generalize the 1-D stability of traveling waves to the $n$-D stability of planar traveling waves, namely, the non-critical planar traveling waves are exponentially stable in the form of $t^{-n/2}e^{-\nu_\tau t}$ and the critical planar traveling waves are algebraic stable in the form of $t^{-n/2}$. The third is to show how the time-delay affects the convergence rates to the non-critical traveling waves. We will give an explicit form of $\nu_\tau = \nu(\tau)$ to show the effect of the time-delay will essentially make the decay rates of the solutions slowly down. In fact, $\nu_0$ is the biggest as $\tau \to 0$ and $\nu_\infty = 0$ is smallest as $\tau \to \infty$. The reason is that, the time-delayed source term remains those old data at time $\tau$, which doesn't decay as fast as the regular source term without the delay, and causes the solution decay slower than the regular case. A similar phenomenon is also observed in [23] on no blow-up occurring for the time-delayed equations.In fact, it is well-known that, for the Cauchy problem of the parabolic equation

$$u_t - \Delta u = u^p, \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}_+, \quad \text{with } u|_{t=0} = u_0(x),$$

the solution will always blow up at a finite time when $1 < p < p^* = 1 + \frac{2}{n}$ (the *Fujita exponent*). However, for the delayed equation

$$u_t - \Delta u = u^p(x, t-\tau), \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}_+, \quad \tau > 0, \quad \text{with } u|_{t=0} = u_0(x,s), \ s \in [-\tau, 0],$$

by the maximum principle

$$\sup_{t \in [k\tau, (k+1)\tau]} \|u(\cdot, t)\|_{L^\infty(\mathbf{R}^n)} \le \|u(\cdot, k\tau)\|_{L^\infty(\mathbf{R}^n)} + \tau \sup_{t \in [k\tau, (k+1)\tau]} \|u(\cdot, t-\tau)\|_{L^\infty(\mathbf{R}^n)},$$

as showed in [23], the corresponding solution globally exists and never blows up for all $p > 0$.

Throughout this paper, $C > 0$ denotes a generic constant, while $C_i > 0$ and $c_i > 0$ $(i = 0, 1, 2, \cdots)$ represent specific constants. $j = (j_1, j_2, \cdots, j_n)$ denotes a multi-index with non-negative integers $j_i \ge 0$ $(i = 1, \cdots, n)$, and $|j| = j_1 + j_2 + \cdots + j_n$. The derivatives for multi-dimensional function are denoted as $\partial_x^j f(x) := \partial_{x_1}^{j_1} \cdots \partial_{x_n}^{j_n} f(x)$. For an $n$-D function $f(x)$, its Fourier transform is defined as

$$\mathcal{F}[f](\eta) = \hat{f}(\eta) := \int_{\mathbf{R}^n} e^{-\mathrm{i}x \cdot \eta} f(x) dx, \qquad \mathrm{i} := \sqrt{-1},$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{\mathrm{i}x \cdot \eta} \hat{f}(\eta) d\eta.$$

Let $I$ be an interval, typically $I = \mathbf{R}^n$. $L^p(I)$ $(p \ge 1)$ is the Lebesque space of the integrable functions defined on $I$, $W^{k,p}(I)$ $(k \ge 0, p \ge 1)$ is the Sobolev space of the $L^p$-functions $f(x)$ defined on the interval $I$ whose derivatives $\partial_x^j f$ with $|j| = k$ also belong to $L^p(I)$, and in particular, we denote $W^{k,2}(I)$ as $H^k(I)$. Further, $L_w^p(I)$ denotes the weighted $L^p$-space for a weight function $w(x) > 0$ with the norm defined as

$$\|f\|_{L_w^p} = \left( \int_I w(x) |f(x)|^p \, dx \right)^{1/p},$$

$W_w^{k,p}(I)$ is the weighted Sobolev space with the norm given by

$$\|f\|_{W_w^{k,p}} = \left( \sum_{|j|=0}^{k} \int_I w(x) \left| \partial_x^j f(x) \right|^p dx \right)^{1/p},$$

and $H_w^k(I)$ is defined with the norm

$$\|f\|_{H_w^k} = \Big( \sum_{|j|=0}^{k} \int_I w(x) \left| \partial_x^j f(x) \right|^2 dx \Big)^{1/2}.$$

Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C^0([0,T],\mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0,T]$, $L^2([0,T],\mathcal{B})$ as the space of the $\mathcal{B}$-valued $L^2$-functions on $[0,T]$. The corresponding spaces of the $\mathcal{B}$-valued functions on $[0,\infty)$ are defined similarly.

Regarding the existence of monotone traveling wavefronts of (1), as showed in [37], it can be proved by the method of upper-lower solutions in a similar way as in [28, 45, 47, 48, 49], and the critical wave (the wave with the minimum speed) can be further confirmed by the semiflow argument developed in [29], see also [15] by Hamel and Nadirashvili for the existence of planar traveling waves and entire solutions to Fisher-KPP equations.

**Proposition 1.1** (Existence of planar traveling waves [15, 37])**.** *Under the conditions $(H_1)$-$(H_3)$, for the time-delay $\tau \geq 0$, there exist a minimum wave speed (also called the critical wave speed) $c_* > 0$ and a corresponding number $\lambda_* = \lambda(c_*) > 0$ satisfying*

$$(10) \qquad F_{c_*}(\lambda_*) = G_{c_*}(\lambda_*), \qquad F'_{c_*}(\lambda_*) = G'_{c_*}(\lambda_*),$$

*where*

$$(11) \qquad F_c(\lambda) = b'(0) e^{\alpha \lambda^2 - \lambda c \tau}, \qquad G_c(\lambda) = c\lambda - D\lambda^2 + d'(0),$$

*and $(c_*, \lambda_*)$ is the tangent point of $F_c(\lambda)$ and $G_c(\lambda)$, namely,*

$$(12) \qquad b'(0) e^{\alpha \lambda_*^2 - \lambda_* c_* \tau} = c_* \lambda_* - D\lambda_*^2 + d'(0),$$

$$(13) \qquad b'(0)(2\alpha\lambda_* - c_*\tau) e^{\alpha \lambda_*^2 - \lambda_* c_* \tau} = c_* - 2D\lambda_*,$$

*such that for any $c \geq c_*$, there exits a monotone traveling wavefront $\phi(x_1 + ct)$ of (6) connecting $u_\pm$ exists, and for any $c < c_*$, no traveling wave $\phi(x_1 + ct)$ exists.*

*In the case of $c > c_*$, there exist two numbers depending on $c$: $\lambda_1 = \lambda_1(c) > 0$ and $\lambda_2 = \lambda_2(c) > 0$ as the solutions to the equation $F_c(\lambda_i) = G_c(\lambda_i)$, i.e.,*

$$(14) \qquad b'(0) e^{\alpha \lambda_i^2 - \lambda_i c \tau} = c\lambda_i - D\lambda_i^2 + d'(0), \quad i = 1, 2,$$

*such that*

$$(15) \qquad F_c(\lambda) < G_c(\lambda) \quad for \ \lambda_1 < \lambda < \lambda_2,$$

*and particularly,*

$$(16) \qquad F_c(\lambda_*) < G_c(\lambda_*) \ with \ \lambda_1 < \lambda_* < \lambda_2.$$

*In the case of $c = c_*$, it holds*

$$(17) \qquad F_{c_*}(\lambda_*) = G_{c_*}(\lambda_*) \ with \ \lambda_1 = \lambda_* = \lambda_2.$$

*When $\xi_1 = x_1 + ct \to -\infty$, for all $c \geq c_*$, the traveling wavefronts $\phi(x_1 + ct)$ converge to $u_- = 0$ exponentially as follows*

$$(18) \qquad |\phi(\xi_1) - u_-| = O(1) e^{-\lambda_1 |\xi_1|},$$

*where $\lambda^1 = \lambda_1(c) > 0$ is given in (14), particularly, $\lambda_1 = \lambda_*$ for $c = c_*$.*

Before stating our main theorems, we define a weight function as

$$
(19) \qquad w(x) = \begin{cases} e^{-\lambda_*(x_1 - x_*)}, & \text{for } x_1 \leq x_*, \\ 1, & \text{for } x_1 > x_*, \end{cases}
$$

where $x_* > 0$ is a sufficiently large number such that, for $x_1 \geq x_* \gg 1$,

$$
(20) \qquad d'(\phi(x_1)) \geq \int_{\mathbf{R}^n} f_\alpha(y) b'(\phi(x_1 - y_1 - c\tau)) dy.
$$

**Theorem 1.2** (Stability of planar traveling waves)**.** *Let $d(u)$ and $b(u)$ satisfy $(H_1)$-$(H_3)$. For a given traveling wave $\phi(x_1 + ct)$ of the equation (1) with $c \geq c_*$ and $\phi(\pm\infty) = u_\pm$, if the initial data satisfy*

$$
(21) \qquad 0 = u_- \leq u_0(s, x) \leq u_+, \quad \text{for } (s, x) \in [-\tau, 0] \times \mathbf{R}^n,
$$

*and the initial perturbation $u_0(s, x) - \phi(x_1 + cs)$ is in $C^1([-\tau, 0], W_w^{2,1}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n))$, then the solution of (1) uniquely exists and satisfies:*

    (i) *When $c > c_*$, the solution $u(t, x)$ converges to the noncritical planar traveling wave $\phi(x + ct)$ exponentially*

$$
(22) \qquad \sup_{x \in \mathbf{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq Ct^{-\frac{n}{2}} e^{-\nu_\tau t}, \quad t > 0
$$

       *for some constant*

$$
0 < \nu_\tau < \max\{\varepsilon_1(\tau)(c_1 - c_3(\tau)), d'(u_+) - b'(u_+)\},
$$

       *where $c_1 := c\lambda_* - D\lambda_*^2 + d'(0)$, $c_3(\tau) := b'(0)e^{\alpha\lambda_*^2 - c\lambda_*\tau}$, and $\varepsilon_1(\tau) \in (0, 1)$ is uniquely determined and is decreasing with respect to the time-delay $\tau$, such that $\varepsilon_1(\tau) \to 0$ as $\tau \to \infty$, and $\varepsilon_1(\tau) \to 1$ as $\tau \to 0$.*

    (ii) *When $c = c_*$, the solution $u(t, x)$ converges to the critical planar traveling wave $\phi(x_1 + c_* t)$ algebraically*

$$
(23) \qquad \sup_{x \in \mathbf{R}^n} |u(t, x) - \phi(x_1 + c_* t)| \leq Ct^{-\frac{n}{2}}, \quad t > 0.
$$

**Remark 1.3.**

    a) *The convergence rates showed in Theorem 1.2 are optimal, when the initial perturbation around the wave is in the weighted $L_w^1(\mathbb{R})$ space.*

    b) *It is noted that, the previous stability results obtained in [38, 36, 37] for 1-D nonlocal time-delayed reaction-diffusion equations, [39, 35] for Nicolson's blowflies equations, [26, 40] for population models with age structure, and [43, 41, 21, 13] for 1-D classical Fisher-KPP equations, all are the special cases of our stability Theorem 1.2. Particularly, our convergence rates are more precise.*

    c) *The conditions $d'(u_+)^2 > b'(0)b'(u_+)$ required in [37] is removed in our present paper.*

When the time-delay $\tau = 0$ and $\alpha \to 0^+$, the time-delayed nonlocal equation (1) reduces to the regular Fisher-KPP equation (4). Assume that the equation (4) is mono-stable, namely, $h(u)$ satisfies

    (H′) There exist $u_- = 0$ and $u_+ > 0$ such that $h(u) \in C^2[0, u_+]$, $h(0) = h(u_+) = 0$, $h'(0) > 0$, $h'(u_+) < 0$, and $h''(u) \leq 0$ for $u \in [0, u_+]$.

From Theorem 1.2, we immediately obtain the exponential stability for the non-critical planar traveling waves and the algebraic stability for the critical planar traveling waves to the regular Fisher-KPP equations (4).

**Corollary 1.4.** *Let $h(u)$ satisfy $(H')$. For a given traveling wave $\phi(x_1 + ct)$ of the equation (4) with $c \geq c_* = 2\sqrt{h'(0)}$ and $\phi(\pm\infty) = u_\pm$, if the initial data satisfy*

(24) $$0 = u_- \leq u_0(x) \leq u_+, \quad \text{for } x \in \mathbf{R}^n,$$

*and the initial perturbation $u_0(x) - \phi(x_1)$ is in $W_w^{2,1}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, then the solution of (4) uniquely exists and satisfies:*

   (i) *When $c > c_*$, the solution $u(t, x)$ converges to the noncritical planar traveling wave $\phi(x + ct)$ exponentially*

(25) $$\sup_{x \in \mathbf{R}^n} |u(t,x) - \phi(x_1 + ct)| \leq Ct^{-\frac{n}{2}} e^{-\nu_0 t}, \quad t > 0$$

   *for some constant*

$$0 < \nu_0 < \max\{c - c_*, |h'(u_+)|\},$$

   *where*

   (ii) *When $c = c_*$, the solution $u(t,x)$ converges to the critical planar traveling wave $\phi(x_1 + c_* t)$ algebraically*

(26) $$\sup_{x \in \mathbf{R}^n} |u(t,x) - \phi(x_1 + c_* t)| \leq Ct^{-\frac{n}{2}}, \quad t > 0.$$

The rest of this paper is organized as follows. In section 2, we will build up some crucial energy decay estimates for the solutions to the linearized nonlocal reaction-diffusion equations, which will be the key for the stability proof. Particularly, we will show that the effect of the time-delay essentially makes the convergence in the case with time-delay much faster than the case without time-delay. In section 3, we will further prove the global asymptotic stability results with a time-exponential decay for the noncritical traveling waves and a time-algebraic decay for the critical traveling wave, respectively. The adopted approach for proof is based on the method developed in [41, 37], but to derive the Green function with time-delay for equation (1) as well as the optimal decay rates of the solutions is main contribution in this paper.

## 2. Linearized Nonlocal Reaction-Diffusion Equations

In this section, we will derive the solution formula for the linear delayed ordinary differential equations, and the formula for the linearized nonlocal reaction-diffusion equations with or without time-delay, as well as their asymptotic behaviors, which will play a key role in the stability proof in section 3.

Now let us introduce the solution formula for linear delayed ODEs as shown in [20].

**Lemma 2.1** ([20]). *Let $z(t)$ be the solution to the following linear time-delayed ODE with time-delay $\tau > 0$*

(27) $$\begin{cases} \dfrac{d}{dt} z(t) + k_1 z(t) = k_2 z(t - \tau) \\ z(s) = z_0(s), \quad s \in [-\tau, 0]. \end{cases}$$

*Then*

(28) $$z(t) = e^{-k_1(t+\tau)} e_\tau^{\bar{k}_2 t} z_0(-\tau) + \int_{-\tau}^0 e^{-k_1(t-s)} e_\tau^{\bar{k}_2(t-\tau-s)} [z_0'(s) + k_1 z_0(s)] ds,$$

*where*

(29) $$\bar{k}_2 := k_2 e^{k_1 \tau},$$

and $e_\tau^{\bar{k}_2 t}$ is the so-called `delayed exponential function` in the form

$$(30) \quad e_\tau^{\bar{k}_2 t} = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \leq t < 0, \\ 1 + \frac{\bar{k}_2 t}{1!}, & 0 \leq t < \tau, \\ 1 + \frac{\bar{k}_2 t}{1!} + \frac{\bar{k}_2^2 (t-\tau)^2}{2!}, & \tau \leq t < 2\tau, \\ \vdots & \vdots \\ 1 + \frac{\bar{k}_2 t}{1!} + \frac{\bar{k}_2^2 (t-\tau)^2}{2!} + \cdots + \frac{\bar{k}_2^m [t-(m-1)\tau]^m}{m!}, & (m-1)\tau \leq t < m\tau, \\ \vdots & \vdots \end{cases}$$

and $e_\tau^{\bar{k}_2 t}$ is the fundamental solution to

$$(31) \quad \begin{cases} \dfrac{d}{dt} z(t) = \bar{k}_2 z(t-\tau) \\ z(s) \equiv 1, \quad s \in [-\tau, 0]. \end{cases}$$

Note that, different from the exponential function $e^{(k_1+k_2)t} = e^{k_1 t} e^{k_2 t}$, we always have

$$e_\tau^{(k_1+k_2)t} \neq e_\tau^{k_1 t} e_\tau^{k_2 t}.$$

Furthermore, we can prove that such a fundamental solution (the delayed exponential function) captures the following asymptotic behavior.

**Lemma 2.2.** *Let $k_1 \geq 0$ and $k_2 \geq 0$. Then the solution $z(t)$ to (27) (or equivalently (28)) satisfies*

$$(32) \quad |z(t)| \leq C_0 e^{-k_1 t} e_\tau^{\bar{k}_2 t},$$

*where*

$$(33) \quad C_0 := e^{-k_1 \tau} |z_0(-\tau)| + \int_{-\tau}^0 e^{k_1 s} |z_0'(s) + k_1 z_0(s)| ds,$$

*and the fundamental solution $e_\tau^{\bar{k}_2 t}$ with $\bar{k}_2 > 0$ to (31) satisfies*

$$(34) \quad e_\tau^{\bar{k}_2 t} \leq C(1+t)^{-\gamma} e^{\bar{k}_2 t}, \quad t > 0$$

*for arbitrary number $\gamma > 0$.*

*Furthermore, when $k_1 \geq k_2 \geq 0$, there exists a constant $0 < \varepsilon_1 < 1$ such that*

$$(35) \quad e^{-k_1 t} e_\tau^{\bar{k}_2 t} \leq C e^{-\varepsilon_1 (k_1 - k_2) t}, \quad t > 0$$

*and the solution $z(t)$ to (27) satisfies*

$$(36) \quad |z(t)| \leq C e^{-\varepsilon_1 (k_1 - k_2) t}, \quad t > 0.$$

**Proof.** For (32), it is easy to see from (28) that

$$\begin{aligned} |z(t)| &= \left| e^{-k_1(t+\tau)} e_\tau^{\bar{k}_2 t} z_0(-\tau) + \int_{-\tau}^0 e^{-k_1(t-s)} e_\tau^{\bar{k}_2(t-\tau-s)} [z_0'(s) + k_1 z_0(s)] ds \right| \\ &= \left| e^{-k_1 \tau} z_0(-\tau) + \int_{-\tau}^0 e^{k_1 s} \frac{e_\tau^{\bar{k}_2(t-\tau-s)}}{e_\tau^{\bar{k}_2 t}} [z_0'(s) + k_1 z_0(s)] ds \right| e^{-k_1 t} e_\tau^{\bar{k}_2 t} \\ &\leq \left\{ e^{-k_1 \tau} |z_0(-\tau)| + \int_{-\tau}^0 e^{k_1 s} |z_0'(s) + k_1 z_0(s)| ds \right\} e^{-k_1 t} e_\tau^{\bar{k}_2 t} \\ (37) \quad &=: C_0 e^{-k_1 t} e_\tau^{\bar{k}_2 t}, \end{aligned}$$

where we used the fact, by the definition of $e_\tau^{\bar{k}_2 t}$ given in (30),

$$0 \leq \frac{e_\tau^{\bar{k}_2(t-\tau-s)}}{e_\tau^{\bar{k}_2 t}} \leq 1 \quad \text{for} \ \ s \in [-\tau, 0].$$

To prove (34), we need to construct an upper-solution to (31). From the assumption we know $\bar{k}_2 > 0$. For an arbitrarily given number $\gamma > 0$, we have

$$\lim_{t \to \infty} \left[ 1 - \frac{\gamma}{\bar{k}_2}(1+t)^{-1} - e^{-\bar{k}_2 \tau}\left( \frac{1+t}{1+t-\tau} \right)^\gamma \right] = 1 - e^{-\bar{k}_2 \tau} > 0.$$

So there exists a large number $t_0 = t_0(\gamma, \bar{k}_2, \tau) = m_0 \tau > 0$, where $m_0 \in \mathbb{N}$ is a positive and large enough integer, such that

$$(38) \qquad 1 - \frac{\gamma}{\bar{k}_2}(1+t)^{-1} - e^{-\bar{k}_2 \tau}\left( \frac{1+t}{1+t-\tau} \right)^\gamma > 0, \quad \text{for} \ \ t \geq t_0 > 0.$$

We now verify that

$$(39) \qquad \bar{z}(t) := C_1(1+t)^{-\gamma}e^{\bar{k}_2 t}, \ \ \text{for } t \geq t_0, \ \ \text{with } C_1 \geq e_\tau^{\bar{k}_2 t_0}$$

is an upper-solution to (31) for $t \in [t_0, \infty)$, namely,

$$(40) \qquad \frac{d}{dt}\bar{z}(t) \geq \bar{k}_2 \bar{z}(t - \tau), \quad \bar{z}(s) \geq e_\tau^{\bar{k}_2 s}, \ \ s \in [t_0 - \tau, t_0].$$

In fact, it holds that

$$\frac{d}{dt}\bar{z}(t) - k_2 \bar{z}(t - \tau)$$
$$= C_1 \bar{k}_2(1+t)^{-\gamma}e^{\bar{k}_2 t} - C_1 \gamma(1+t)^{-\gamma-1}e^{\bar{k}_2 t} - \bar{k}_2 C_1(1+t-\tau)^{-\gamma}e^{\bar{k}_2(t-\tau)}$$
$$= C_1 \bar{k}_2(1+t)^{-\gamma}e^{\bar{k}_2 t}\left[ 1 - \frac{\gamma}{\bar{k}_2}(1+t)^{-1} - e^{-\bar{k}_2 \tau}\left( \frac{1+t}{1+t-\tau} \right)^\gamma \right]$$

$$(41) \qquad \geq 0, \quad \text{for } t \geq t_0.$$

So, by the comparison principle to the linear delayed ODE, the upper-solution $\bar{z}(t)$ is always greater than the fundamental solution $e_\tau^{\bar{k}_2 t}$ to (31) in $[t_0, \infty)$, namely,

$$(42) \qquad\qquad e_\tau^{\bar{k}_2 t} \leq C_1(1+t)^{-\gamma}e^{\bar{k}_2 t}, \ \ t \geq t_0.$$

On the other hand, for the bounded interval $[0, t_0]$, we can always have

$$(43) \qquad\qquad e_\tau^{\bar{k}_2 t} \leq C_1(1+t)^{-\gamma}e^{\bar{k}_2 t}, \ \ t \in [0, t_0]$$

by selecting a large number $C_1 > 0$. Thus, combing (42) and (43) gives (34).

To prove (35) and (36), we are going to carry it out in two cases. When $k_1 > k_2 \geq 0$, we can similarly construct a pair of upper and lower solutions to (27) to prove (35) and (36). Let $\bar{z}(t) := M e^{(k_1-k_2)\tau}e^{-\varepsilon_1(k_1-k_2)t}$, where $M := \max_{s \in [-\tau, 0]} |z_0(s)|$. It can be verified that $\bar{z}|_{t=s} \geq |z_0(s)|$ and

$$\frac{d}{dt}\bar{z}(t) + k_1 \bar{z}(t) - k_2 \bar{z}(t-\tau) = M e^{(k_1-k_2)\tau}e^{-\varepsilon_1(k_1-k_2)t}[(1-\varepsilon_1)k_1 + \varepsilon_1 k_2 - k_2 e^{\varepsilon_1(k_1-k_2)\tau}] = 0,$$

where $0 < \varepsilon_1 < 1$ is uniquely determined by

$$(44) \qquad\qquad (1 - \varepsilon_1)k_1 + \varepsilon_1 k_2 - k_2 e^{\varepsilon_1(k_1-k_2)\tau} = 0.$$

So, we have proved that $\bar{z}(t) := M e^{(k_1-k_2)\tau}e^{-\varepsilon_1(k_1-k_2)t}$ is an upper solution of (27) such that

$$(45) \qquad\qquad z(t) \leq M e^{(k_1-k_2)\tau}e^{-\varepsilon_1(k_1-k_2)t} \quad \text{for } t > 0.$$

Let $\tilde{z}(t) := -z(t)$, then it satisfies

$$\frac{d}{dt}\tilde{z}(t) + k_1\tilde{z}(t) = k_2\tilde{z}(t - \tau)$$

As showed before, we can verify

$$(46) \qquad z(t) \geq -Me^{(k_1-k_2)\tau}e^{-\varepsilon_1(k_1-k_2)t} \quad \text{for } t > 0.$$

Thus, (45) and (46) together imply

$$(47) \qquad |z(t)| \leq Ce^{-\varepsilon_1(k_1-k_2)t} \quad \text{for } k_1 > k_2, \ t > 0.$$

In order to prove (35) in the case of $k_1 > k_2$, let us particularly select the initial data $z_0(s) \equiv 1$, and apply (36) and (28), namely, the corresponding solution is

$$z(t) = e^{-k_1(t+\tau)}e_\tau^{\bar{k}_2 t} + k_1\int_{-\tau}^0 e^{-k_1(t-s)}e_\tau^{\bar{k}_2(t-\tau-s)}ds > 0,$$

we then have

$$e^{-k_1(t+\tau)}e_\tau^{\bar{k}_2 t} \leq e^{-k_1(t+\tau)}e_\tau^{\bar{k}_2 t} + k_1\int_{-\tau}^0 e^{-k_1(t-s)}e_\tau^{\bar{k}_2(t-\tau-s)}ds = z(t) \leq Ce^{-\varepsilon_1(k_1-k_2)t}.$$

This implies

$$(48) \qquad e^{-k_1 t}e_\tau^{\bar{k}_2 t} \leq Ce^{-\varepsilon_1(k_1-k_2)t} \quad \text{for } k_1 > k_2.$$

When $k_1 = k_2$, let $M := \max_{s\in[-\tau,0]} z_0(s)$ and $m := \min_{s\in[-\tau,0]} z_0(s)$. It is easy to see that $\bar{z}(t) = M$ and $\underline{z}(t) = m$ are the upper-solution and lower-solution of the equation (27), respectively. So,

$$m \leq z(t) \leq M.$$

This implies

$$(49) \qquad |z(t)| \leq C = Ce^{-\varepsilon_1(k_1-k_2)t} \quad \text{for } k_1 = k_2$$

and

$$(50) \qquad e^{-k_1 t}e_\tau^{\bar{k}_2 t} \leq C = Ce^{-\varepsilon_1(k_1-k_2)t} \quad \text{for } k_1 = k_2.$$

Combining (47), (48), (49) and (50), we prove (35) and (36) for $k_1 \geq k_2$ with $0 < \varepsilon_1 < 1$. Thus, the proof for this lemma is complete. $\square$

Next, we consider the following nonlocal linearized time-delayed reaction-diffusion equation (which will be derived in section 3 for the proof of stability of traveling wavefronts)

$$(51) \qquad \begin{cases} \dfrac{\partial v}{\partial t} + \displaystyle\sum_{i=1}^n c_{0i}\dfrac{\partial v}{\partial x_i} - D\Delta v + c_1 v \\ \qquad = c_2\displaystyle\int_{\mathbf{R}^n} f_\alpha(y)e^{-\lambda_*(y_1+c\tau)}v(t-\tau, x-y-c\tau\mathbf{e}_1)dy \\ v(s,x) = v_0(s,x), \quad s\in[-\tau,0], \ x\in\mathbf{R}^n \end{cases}$$

for some given constant coefficients $c_{0i}$, $c_1$ and $c_2$. Here

$$x - y - c\tau\mathbf{e}_1 = (x_1 - y_1 - c\tau, x_2 - y_2, \cdots, x_n - y_n).$$

Now we are going to derive its solution formula as well as the asymptotic behavior of the solution. By taking Fourier transform to (51), we have

$$(52) \quad \frac{d\hat{v}}{dt} + A(\eta)\hat{v} = \mathcal{F}\left[c_2\int_{\mathbf{R}^n} f_\alpha(y)e^{-\lambda_*(y_1+c\tau)}v(t-\tau, x-y-c\tau\mathbf{e}_1)dy\right](t-\tau,\eta),$$

where

$$(53) \qquad A(\eta) = D|\eta|^2 + c_1 + \mathrm{i} \sum_{i=1}^{n} c_{0i}\eta_i, \quad \text{with} \ \ |\eta|^2 = \sum_{i=1}^{n} \eta_i^2.$$

Note that,

$$\mathcal{F}\Big[c_2 \int_{\mathbf{R}^n} f_\alpha(y) e^{-\lambda_*(y_1+c\tau)} v(t-\tau, x-y-c\tau\mathbf{e}_1) dy\Big](t-\tau,\eta)$$

$$= c_2 \int_{\mathbf{R}^n} e^{-\mathrm{i}x\cdot\eta} \Big( \int_{\mathbf{R}^n} f_\alpha(y) e^{-\lambda_*(y_1+c\tau)} v(t-\tau, x-y-c\tau\mathbf{e}_1) dy \Big) dx$$

$$= c_2 \int_{\mathbf{R}^n} f_\alpha(y) e^{-\lambda_*(y_1+c\tau)} \Big( \int_{\mathbf{R}^n} e^{-\mathrm{i}x\cdot\eta} v(t-\tau, x-y-c\tau\mathbf{e}_1) dx \Big) dy$$

$$= c_2 \int_{\mathbf{R}^n} f_\alpha(y) e^{-\lambda_*(y_1+c\tau)} \Big( \int_{\mathbf{R}^n} e^{-\mathrm{i}(x+y+c\tau\mathbf{e}_1)\cdot\eta} v(t-\tau, x) dx \Big) dy$$

$$= c_2 \int_{\mathbf{R}^n} f_\alpha(y) e^{-\lambda_*(y_1+c\tau)} e^{-\mathrm{i}(y+c\tau\mathbf{e}_1)\cdot\eta} \Big( \int_{\mathbf{R}^n} e^{-\mathrm{i}x\cdot\eta} v(t-\tau, x) dx \Big) dy$$

$$= \Big(c_2 \int_{\mathbf{R}^n} f_\alpha(y) e^{-\lambda_*(y_1+c\tau)} e^{-\mathrm{i}(y+c\tau\mathbf{e}_1)\cdot\eta} dy \Big) \hat{v}(t-\tau,\eta)$$

$$(54) \qquad = B(\eta)\hat{v}(t-\tau,\eta),$$

where

$$(55) \qquad B(\eta) := c_2 \int_{\mathbf{R}^n} f_\alpha(y) e^{-\lambda_*(y_1+c\tau)} e^{-\mathrm{i}(y+c\tau\mathbf{e}_1)\cdot\eta} dy.$$

Substituting (54) to (52), we have

$$(56) \qquad \frac{d\hat{v}}{dt} + A(\eta)\hat{v} = B(\eta)\hat{v}(t-\tau,\eta), \quad \text{with} \ \hat{v}(s,\eta) = \hat{v}_0(s,\eta), \ s \in [-\tau, 0].$$

Solving (56) by the solution formula (28), we obtain

$$\hat{v}(t,\eta) = e^{-A(\eta)(t+\tau)} e_\tau^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau,\eta)$$

$$(57) \qquad \qquad + \int_{-\tau}^{0} e^{-A(\eta)(t-s)} e_\tau^{\mathcal{B}(\eta)(t-\tau-s)} \Big[ \frac{d}{ds}\hat{v}_0(s,\eta) + A(\eta)\hat{v}_0(s,\eta) \Big] ds,$$

where

$$(58) \qquad \mathcal{B}(\eta) := B(\eta) e^{A(\eta)\tau}.$$

Then, by taking the inverse Fourier transform to (57), we get

$$v(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{\mathrm{i}x\cdot\eta} e^{-A(\eta)(t+\tau)} e_\tau^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau,\eta) d\eta$$

$$+ \int_{-\tau}^{0} \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{\mathrm{i}x\cdot\eta} e^{-A(\eta)(t-s)} e_\tau^{\mathcal{B}(\eta)(t-\tau-s)}$$

$$(59) \qquad \qquad \times \Big[ \frac{d}{ds}\hat{v}_0(s,\eta) + A(\eta)\hat{v}_0(s,\eta) \Big] d\eta ds,$$

and its derivatives, for a multi-index $j = (j_1, \cdots, j_n)$ with nonnegative integers $j_i$ and $|j| = \sum_{i=1}^n j_i$, can be expressed as

$$
\partial_x^j v(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{\mathrm{i}x \cdot \eta} \Big( \prod_{i=1}^n (\mathrm{i}\eta_i)^{j_i} \Big) e^{-A(\eta)(t+\tau)} e_\tau^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau, \eta) d\eta
$$

$$
+ \int_{-\tau}^0 \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{\mathrm{i}x \cdot \eta} \Big( \prod_{i=1}^n (\mathrm{i}\eta_i)^{j_i} \Big) e^{-A(\eta)(t-s)} e_\tau^{\mathcal{B}(\eta)(t-\tau-s)}
$$

$$
(60) \qquad\qquad \times \Big[ \frac{d}{ds} \hat{v}_0(s, \eta) + A(\eta)\hat{v}_0(s, \eta) \Big] d\eta ds.
$$

Now we are going to derive the asymptotic behavior of $v(t, x)$.

**Theorem 2.3** (Decay rates). *Let* $v_0 \in C^1([-\tau, 0]; W^{2,1}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n))$ *and*

$$
(61) \qquad\qquad c_3 := c_2 \int_{\mathbf{R}^n} f_\alpha(y) e^{-\lambda_*(y_1 + c\tau)} dy > 0.
$$

*Then, for* $c_1 \geq c_3$, *there exists a constant* $0 < \varepsilon_1 < 1$ *such that the solution of the linearized equation* (51) *satisfies*

$$
(62) \qquad\qquad \|\partial_x^j v(t)\|_{L^2(\mathbf{R}^n)} \leq C t^{-\frac{n}{4} - \frac{|j|}{2}} e^{-\varepsilon_1(c_1 - c_3)t}, \quad t > 0,
$$

$$
(63) \qquad\qquad \|\partial_x^j v(t)\|_{L^\infty(\mathbf{R}^n)} \leq C t^{-\frac{n}{2} - \frac{|j|}{2}} e^{-\varepsilon_1(c_1 - c_3)t}, \quad t > 0.
$$

**Proof**. Denote

$$
(64) \qquad I_1(t, \eta) := \Big( \prod_{i=1}^n (\mathrm{i}\eta_i)^{j_i} \Big) e^{-A(\eta)(t+\tau)} e_\tau^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau, \eta),
$$

$$
I_2(t - s, \eta) := \Big( \prod_{i=1}^n (\mathrm{i}\eta_i)^{j_i} \Big) e^{-A(\eta)(t-s)} e_\tau^{\mathcal{B}(\eta)(t-\tau-s)}
$$

$$
(65) \qquad\qquad\qquad \times \Big[ \frac{d}{ds} \hat{v}_0(s, \eta) + A(\eta)\hat{v}_0(s, \eta) \Big].
$$

Then, (60) is reduced to

$$
(66) \qquad \partial_x^j v(t, x) = \mathcal{F}^{-1}[I_1](t, x) + \int_{-\tau}^0 \mathcal{F}^{-1}[I_2](t - s, x) ds.
$$

So, by using Parseval's equality, we have

$$
\|\partial_x^j v(t)\|_{L^2(\mathbf{R}^n)} \leq \|\mathcal{F}^{-1}[I_1](t)\|_{L^2(\mathbf{R}^n)} + \int_{-\tau}^0 \|\mathcal{F}^{-1}[I_2](t - s)\|_{L^2(\mathbf{R}^n)} ds
$$

$$
(67) \qquad\qquad = \|I_1(t)\|_{L^2(\mathbf{R}^n)} + \int_{-\tau}^0 \|I_2(t - s)\|_{L^2(\mathbf{R}^n)} ds.
$$

Note that, from (53) and (55), we have

$$
(68) \qquad |e^{-A(\eta)(t+\tau)}| = e^{-(D|\eta|^2 + c_1)(t+\tau)} = e^{-k_1(t+\tau)}
$$

by defining

$$
(69) \qquad\qquad k_1 := D|\eta|^2 + c_1,
$$

and

$$
(70) \qquad |B(\eta)| \leq c_2 \int_{\mathbf{R}^n} f_\alpha(y) e^{-\lambda_*(y_1 + c\tau)} dy = c_3 =: k_2,
$$

and

$$
(71) \qquad |\mathcal{B}(\eta)| = |B(\eta) e^{A(\eta)\tau}| \leq c_3 e^{(D|\eta|^2 + c_1)\tau} = k_2 e^{k_1 \tau} =: \bar{k}_2,
$$

and further

(72)
$$|e_\tau^{\mathcal{B}(\eta)t}| \le e_\tau^{\bar{k}_2 t}.$$

By the definition of Fourier transform, we have

(73)
$$\sup_{\eta \in \mathbf{R}^n} |\hat{v}_0(-\tau, \eta)| \le \int_{\mathbf{R}^n} |v_0(-\tau, x)| dx = \|v_0(-\tau)\|_{L^1(\mathbf{R}^n)},$$

then, we obtain

$$\|I_1(t)\|_{L^2(\mathbf{R}^n)}$$
$$= \Big[ \int_{\mathbf{R}^n} \Big| \Big( \prod_{i=1}^n (\mathrm{i}\eta_i)^{j_i} \Big) e^{-A(\eta)(t+\tau)} e_\tau^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau, \eta) \Big|^2 d\eta \Big]^{1/2}$$

(74)
$$\le \Big[ \int_{\mathbf{R}^n} \Big( \prod_{i=1}^n |\eta_i|^{2j_i} \Big) \Big( e^{-k_1(t+\tau)} e_\tau^{\bar{k}_2 t} \Big)^2 |\hat{v}_0(-\tau, \eta)|^2 d\eta \Big]^{1/2}.$$

Noting (69) for $k_1 = D|\eta|^2 + c_1$, (70) for $k_2 = c_3$ and (71) for $\bar{k}_2 = k_2 e^{k_1 \tau}$, and $k_1 = D|\eta|^2 + c_1 \ge c_3 = k_2$ due to $c_1 \ge c_3$, and using (35) in Lemma 2.2, we get

(75)
$$e^{-k_1 t} e_\tau^{\bar{k}_2 t} \le C e^{-\varepsilon_1(k_1 - k_2)t} = C e^{-\varepsilon_1(D|\eta|^2 + c_1 - c_3)t}$$

for some constant $1 > \varepsilon_1 > 0$.

Thus, applying (75) into (74), we obtain

$$\|I_1(t)\|_{L^2(\mathbf{R}^n)}$$
$$\le C \Big( \sup_{\eta \in \mathbf{R}^n} |\hat{v}_0(-\tau, \eta)|^2 \cdot e^{-2\varepsilon_1(c_1 - c_3)t} \prod_{i=1}^n \int_{\mathbf{R}} |\eta_i|^{2j_i} e^{-2\varepsilon_1 D\eta_i^2 t} d\eta_i \Big)^{1/2}$$
$$\le C \Big( \sup_{\eta \in \mathbf{R}^n} |\hat{v}_0(-\tau, \eta)| \Big) e^{-\varepsilon_1(c_1 - c_3)t} \Big( \prod_{i=1}^n \int_{\mathbf{R}} |\eta_i|^{2j_i} e^{-2\varepsilon_1 D\eta_i^2 t} d\eta_i \Big)^{1/2}$$

(76)
$$\le C t^{-\frac{n}{4} - \frac{|j|}{2}} e^{-\varepsilon_1(c_1 - c_3)t} \|v_0(-\tau)\|_{L^1(\mathbf{R}^n)},$$

where we used

$$\Big( \prod_{i=1}^n \int_{\mathbf{R}} |\eta_i|^{2j_i} e^{-2\varepsilon_1 D\eta_i^2 t} d\eta_i \Big)^{1/2}$$
$$= \Big( \prod_{i=1}^n \frac{1}{(\sqrt{t})^{2j_i + 1}} \int_{\mathbf{R}} |\zeta_i|^{2j_i} e^{-2\varepsilon_1 D\zeta_i^2} d\zeta \Big)^{1/2} \qquad \text{[by substituting } \zeta := \sqrt{t}\eta \text{]}$$
$$\le C \Big( \prod_{i=1}^n t^{-j_i - \frac{1}{2}} \Big)^{1/2} = C t^{-\frac{|j|}{2} - \frac{n}{4}}.$$

Again, by using the property of Fourier transform, we have

$$|(\mathrm{i}\eta_i)^{j_i} \hat{v}_0| = |\widehat{\partial_{x_i}^{j_i} v_0}| \le \int_{\mathbf{R}^n} |\partial_{x_i}^{j_i} v_0| dx = \|\partial_{x_i}^{j_i} v_0\|_{L^1(\mathbf{R}^n)},$$

and

$$\sup_{\eta \in \mathbf{R}^n} |A(\eta) \hat{v}_0(s, \eta)| = \sup_{\eta \in \mathbf{R}^n} \Big| \Big( D|\eta|^2 + c_1 + \mathrm{i} \sum_{i=1}^n c_{0i}\eta_i \Big) \hat{v}_0(s, \eta) \Big|$$

(77)
$$\le C \|v_0(s)\|_{W^{2,1}(\mathbf{R}^n)}.$$

Thus, in a similar way, we can prove

$$\|I_2(t-s)\|_{L^2(\mathbf{R}^n)}$$

$$= \Big[ \int_{\mathbf{R}^n} \Big| \Big( \prod_{i=1}^n (\mathrm{i}\eta_i)^{j_i} \Big) e^{-A(\eta)(t-s)} e_\tau^{\mathcal{B}(\eta)(t-\tau-s)}$$

$$\times [\partial_t \hat{v}_0(s,\eta) + A(\eta)\hat{v}_0(s,\eta)] \Big|^2 d\eta \Big]^{1/2}$$

$$\leq \Big[ \int_{\mathbf{R}^n} \Big( \prod_{i=1}^n |\eta_i|^{2j_i} \Big) e^{-2(D|\eta|^2+c_1)(t-s)} (e_\tau^{\bar{k}_2(t-\tau-s)})^2$$

$$\times |\partial_t \hat{v}_0(s,\eta) A(\eta)\hat{v}_0(s,\eta)|^2 d\eta \Big]^{1/2}$$

$$\leq C \Big( \sup_{\eta \in \mathbf{R}^n} |\partial_t \hat{v}_0(s,\eta) + A(\eta)\hat{v}_0(s,\eta)| \Big) e^{-\varepsilon_1(c_1-c_3)(t-s)}$$

$$\times \Big( \prod_{i=1}^n \int_{\mathbf{R}} |\eta_i|^{2j_i} e^{-2\varepsilon_1 D\eta_i^2(t-s)} ]d\eta_i \Big)^{1/2}$$

$$(78) \qquad \leq C(t-s)^{-\frac{n}{4}-\frac{|j|}{2}} e^{-\varepsilon_1(c_1-c_3)(t-s)} [\|v_{0s}'(s)\|_{L^1(\mathbf{R}^n)} + \|v_0(s)\|_{W^{2,1}(\mathbf{R}^n)}].$$

Substituting (76) and (78) to (67), we immediately obtain (62).

By the same manner, we can similarly prove (63). The detail is omitted. Thus, the proof for this lemma is complete. □

**Remark 2.4.** *In* [37], *the solution to the linear reaction-diffusion equation with time-delay*

$$\frac{\partial v}{\partial t} + k_1 \frac{\partial v}{\partial \xi} - D \frac{\partial^2 v}{\partial \xi^2} = \varepsilon b'(0) e^{k_2\tau} \int_{\mathbf{R}} f_\alpha(y) e^{-\lambda_*(y+c_*\tau)} v(t-\tau, \xi - y - c_*\tau) dy$$

*is given in the integral form of the regular heat equation (no time-delay for the Green function):*

$$v(\xi, t) = \int_{\mathbf{R}} G(t, \xi - \zeta) v_0(0, \zeta) d\zeta$$

$$+ \varepsilon b'(0) e^{k_2\tau} \int_0^t \int_{\mathbf{R}} G(t-s, \xi - \zeta)$$

$$\times \int_{\mathbf{R}} f_\alpha(y) e^{-\lambda_*(y+c_*\tau)} v(t-\tau, \xi - y - c_*\tau) dy d\zeta ds$$

*with* $G(t, \xi - \zeta) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(\xi-\zeta+k_1 t)^2}{4Dt}}$, *which causes the iteration of the solution* $v(t, \xi)$ *in* $[k\tau, (k+1)\tau]$ *step by step with an increasing boundedness like* $C^k(1+t)^{-1/2}$ *in the proof of Lemma 3.7. In order to fix such a gap, here we derive the integral formula of the solution as (51) involving the time-delay* $\tau$, *and obtain the optimal rates* $O(t^{-\frac{n}{2}})$ *for* $c_1 = c_3$, *which is just the case of critical waves as showed in (107) and (108) below.*

## 3. Global Stability for Planar Traveling Waves

It is known that the existence and uniqueness of the solution to (1) can been proved via the standard energy method with continuity-extension method (c.f., [39, 38]) or the theory of abstract functional differential equations [34], so we omit the details here. The main purpose in this section is to prove the stability Theorem 1.2 for all traveling waves including the critical traveling waves with time-delay or not.

Let $c \geq c_*$ and the initial data $u_0(s,x)$ be such that $0 = u_- \leq u_0(s,x) \leq u_+$ for $(s,x) \in [-\tau, 0] \times R$ and $\tau \geq 0$, and define

$$
(79) \quad \begin{cases} U_0^+(s,x) = \max\{u_0(s,x), \phi(x_1 + cs)\}, \\ U_0^-(s,x) = \min\{u_0(s,x), \phi(x_1 + cs)\}, \end{cases} \quad \text{for } (s,x) \in [-\tau, 0] \times \mathbf{R}^n,
$$

which implies

$$(80) \quad 0 = u_- \leq U_0^-(s,x) \leq u_0(s,x) \leq U_0^+(s,x) \leq u_+ \quad \text{for } (s,x) \in [-\tau, 0] \times \mathbf{R}^n,$$

$$(81) \quad 0 = u_- \leq U_0^-(s,x) \leq \phi(x_1 + cs) \leq U_0^+(s,x) \leq u_+ \quad \text{for } (s,x) \in [-\tau, 0] \times \mathbf{R}^n.$$

Note that, the initial data $U_0^{\pm}(s,x)$ are piecewise continuous, and don't have a good enough regularity, which may also cause the absence of regularity for the corresponding solutions. In order to overcome such a shortening, instead of these initial data, we choose two smooth functions as the new initial data:

$$(82) \quad \mathcal{U}_0^+(s,x) \text{ is smooth such that } U_0^+(s,x) \leq \mathcal{U}_0^+(s,x) \leq u_+,$$

$$(83) \quad \mathcal{U}_0^-(s,x) \text{ is smooth such that } u_- \leq \mathcal{U}_0^-(s,x) \leq U_0^-(s,x).$$

Let $U^+(t,x)$ and $U^-(t,x)$ be the corresponding solutions of (1) with the initial data $\mathcal{U}_0^+(s,x)$ and $\mathcal{U}_0^-(s,x)$, respectively, that is,

$$(84) \quad \frac{\partial U^{\pm}}{\partial t} - D\sum_{i=1}^{n} \frac{\partial^2 U^{\pm}}{\partial x_i^2} + d(U^{\pm}) = \int_{\mathbf{R}^n} f_\alpha(y) b(U^{\pm}(t-\tau, x-y)) dy,$$

$$(85) \quad U^{\pm}(s,x) = \mathcal{U}_0^{\pm}(s,x), \quad x \in \mathbf{R}^n, \ s \in [-\tau, 0].$$

By similar arguments as in [26, 30, 35, 36] or the abstract results in [34], it easily follows that equation (1) admits the comparison principle. Thus, we have

$$(86) \quad u_- \leq U^-(t,x) \leq u(t,x) \leq U^+(t,x) \leq u_+ \quad \text{for } (t,x) \in R_+ \times \mathbf{R}^n,$$

$$(87) \quad u_- \leq U^-(t,x) \leq \phi(x_1 + ct) \leq U^+(t,x) \leq u_+ \quad \text{for } (t,x) \in R_+ \times \mathbf{R}^n.$$

In what follows, we are going to complete the proof for the stability in three steps.

**Step 1. The convergence of $U^+(t,x)$ to $\phi(x_1 + ct)$**

For any given $c \geq c_*$, let $\xi = (\xi_1, \xi_2, \cdots, \xi_n) := x + ct\mathbf{e}_1 = (x_1 + ct, x_2, \cdots, x_n) \in \mathbf{R}^n$ and

$$(88) \quad V(t, \xi) := U^+(t,x) - \phi(x_1 + ct), \quad V_0(s, \xi) := \mathcal{U}_0^+(s,x) - \phi(x_1 + cs).$$

It follows from (86) and (87) that

$$(89) \quad V(t, \xi) \geq 0 \quad \text{and} \quad V_0(s, \xi) \geq 0.$$

We see from (1) that $V(t, \xi)$ satisfies (by linearizing it around 0)

$$\frac{\partial V}{\partial t} + c\frac{\partial V}{\partial \xi_1} - D\Delta V + d'(0)V$$

$$- b'(0)\int_{\mathbf{R}^n} f_\alpha(y) V(t-\tau, \xi - y - c\tau\mathbf{e}_1) dy$$

$$= -Q_1(t,\xi) + \int_{\mathbf{R}^n} f_\alpha(y) Q_2(t-\tau, \xi - y - c\tau) dy + [d'(0) - d'(\phi(\xi_1))]V$$

$$+ \int_{\mathbf{R}^n} f_\alpha(y)[b'(\phi(\xi_1 - y_1 - c\tau) - b'(0)]V(t-\tau, \xi - y - c\tau\mathbf{e}_1) dy$$

$$(90) \quad =: I_1(t,\xi) + I_2(t,\xi) + I_3(t,\xi) + I_4(t,\xi),$$

with the initial data

(91) $$V(s, \xi) = V_0(s, \xi), \ s \in [-\tau, 0],$$

where

(92) $$Q_1(t, \xi) = d(\phi + V) - d(\phi) - d'(\phi)V$$

with $\phi = \phi(\xi_1) = \phi(x_1 + ct)$ and $V = V(t, \xi)$, and

(93) $$Q_2(t - \tau, \xi - y - c\tau\mathbf{e}_1) = b(\phi + V) - b(\phi) - b'(\phi)V$$

with $\phi = \phi(\xi_1 - y_1 - c\tau)$ and $V = V(t - \tau, \xi - y - c\tau\mathbf{e}_1)$. Here $I_i(t, \xi), i = 1, 2, 3, 4$, denotes the $i$-th term in the right-side of line above (90).

From (H$_3$), i.e., $d''(u) \geq 0$ and $b''(u) \leq 0$, applying Taylor formula to (92) and (93), we immediately have

$$Q_1(t, \xi) \geq 0 \ \text{ and } \ Q_2(t - \tau, \xi - y - c\tau\mathbf{e}_1) \leq 0,$$

which implies

(94) $$I_1(t, \xi) \leq 0 \ \text{ and } \ I_2(t, \xi) \leq 0.$$

From (H$_3$) again, since $d'(\phi)$ is increasing and $b'(\phi)$ is decreasing, then $d'(0) - d'(\phi(\xi_1)) \leq 0$ and $b'(\phi(\xi_1 - y_1 - c\tau)) - b'(0) \leq 0$, which imply, with $V \geq 0$,

(95) $$I_3(t, \xi) \leq 0 \ \text{ and } \ I_4(t, \xi) \leq 0.$$

Thus, applying (94) and (95) to (90), we obtain

(96) $$\frac{\partial V}{\partial t} + c\frac{\partial V}{\partial \xi_1} - D\Delta V + d'(0)V - b'(0)\int_{\mathbf{R}^n} f_\alpha(y)V(t - \tau, \xi - y - c\tau\mathbf{e}_1)dy \leq 0.$$

Let $\bar{V}(t, \xi)$ be the solution of the following equation with the same initial data $V_0(s, \xi)$:

(97) $$\begin{cases} \dfrac{\partial \bar{V}}{\partial t} + c\dfrac{\partial \bar{V}}{\partial \xi_1} - D\Delta\bar{V} + d'(0)\bar{V} \\ \quad - b'(0)\displaystyle\int_{\mathbf{R}^n} f_\alpha(y)\bar{V}(t - \tau, \xi - y - c\tau\mathbf{e}_1)dy = 0, \quad (t, \xi) \in R_+ \times \mathbf{R}^n, \\ \bar{V}(s, \xi) = V_0(s, \xi), \quad s \in [-\tau, 0], x \in \mathbf{R}^n. \end{cases}$$

By the comparison principle [36], we have

(98) $$0 \leq V(t, \xi) \leq \bar{V}(t, \xi), \quad \text{for } (t, \xi) \in R_+ \times \mathbf{R}^n.$$

Let

(99) $$v(t, \xi) := e^{-\lambda_*(\xi_1 - x_*)}\bar{V}(t, \xi).$$

From (97), $v(t, \xi)$ satisfies

(100) $$\frac{\partial v}{\partial t} + c_0\frac{\partial v}{\partial \xi_1} - D\Delta v + c_1 v$$
$$= c_2\int_{\mathbf{R}^n} f_\alpha(y)e^{-\lambda_*(y_1 + c\tau)}v(t - \tau, \xi - y - c\tau\mathbf{e}_1)dy,$$

where

(101) $$c_0 := c - 2D\lambda_*, \quad c_1 := c\lambda_* - D\lambda_*^2 + d'(0) > 0, \ \text{ and } \ c_2 := b'(0).$$

When $\tau = 0$, then (100) is reduced to

(102) $$\frac{\partial v}{\partial t} + c_0\frac{\partial v}{\partial \xi_1} - D\Delta v + c_1 v = c_2\int_{\mathbf{R}^n} f_\alpha(y)e^{-\lambda_* y_1}v(t, \xi - y)dy.$$

For $\tau > 0$, applying Theorem 2.3, we obtain the following decay rates:

$$\|v(t)\|_{L^\infty(\mathbf{R}^n)} \leq Ct^{-\frac{n}{2}}e^{-\varepsilon_1(c_1-c_3)t}, \tag{103}$$

where $c_3$ is defined in (61), which can be directly calculated as, by using the property (8),

$$
\begin{aligned}
c_3 &= b'(0)\int_{\mathbf{R}^n} f_\alpha(y)e^{-\lambda_*(y_1+c\tau)}dy \\
&= b'(0)\int_{\mathbf{R}} f_{\alpha 1}(y_1)e^{-\lambda_*(y_1+c\tau)}dy_1 \\
&= b'(0)e^{\alpha\lambda_*^2-\lambda_*c\tau} > 0. \tag{104}
\end{aligned}
$$

When $c > c_*$, namely, the wave $\phi(x_1+ct)$ is non-critical, from (16) in Proposition 1.1, we realize

$$c_1 := c\lambda_* - D\lambda_*^2 + d'(0) = G_c(\lambda_*) > F_c(\lambda_*) = b'(0)e^{\alpha\lambda_*^2-\lambda_*c\tau} := c_3. \tag{105}$$

Thus, (103) immediately implies the following exponential decay

$$\|v(t)\|_{L^\infty(\mathbf{R}^n)} \leq Ct^{-\frac{n}{2}}e^{-\varepsilon_1(c_1-c_3)t}. \tag{106}$$

When $c = c_*$, namely, the wave $\phi(x_1+c_*t)$ is critical, from (17) in Proposition 1.1, we realize

$$c_1 := c_*\lambda_* - D\lambda_*^2 + d'(0) = G_{c_*}(\lambda_*) = F_{c_*}(\lambda_*) = b'(0)e^{\alpha\lambda_*^2-\lambda_*c_*\tau} := c_3. \tag{107}$$

Then, from (103), we immediately obtain the following algebraic decay

$$\|v(t)\|_{L^\infty(\mathbf{R}^n)} \leq Ct^{-\frac{n}{2}}. \tag{108}$$

Since $V(t,\xi) \leq \bar{V}(t,\xi) = e^{\lambda_*(\xi_1-x_*)}v(t,\xi)$, and $0 < e^{\lambda_*(\xi_1-x_*)} \leq 1$ for $\xi_1 \in (-\infty, x_*]$, we immediately obtain the following decay for $V$.

**Lemma 3.1.** *It holds that:*

(i) *when $c > c_*$, then*

$$\|V(t)\|_{L^\infty((-\infty,x_*]\times\mathbf{R}^{n-1})} \leq Ct^{-\frac{n}{2}}e^{-\varepsilon_1(c_1-c_3)t}, \quad t > 0; \tag{109}$$

(ii) *when $c = c_*$,*

$$\|V(t)\|_{L^\infty((-\infty,x_*]\times\mathbf{R}^{n-1})} \leq Ct^{-\frac{n}{2}}, \quad t > 0. \tag{110}$$

Next we prove the decay rate for $V(t,\xi)$ in $[x_*, \infty) \times \mathbf{R}^{n-1}$.

**Lemma 3.2.** *It holds that*

$$\|V(t)\|_{L^\infty([x_*,\infty)\times\mathbf{R}^{n-1})} \leq Ct^{-\frac{n}{2}}e^{-\nu t}, \quad for \ c > c_*, \tag{111}$$

$$\|V(t)\|_{L^\infty([x_*,\infty)\times\mathbf{R}^{n-1})} \leq Ct^{-\frac{n}{2}}, \quad for \ c = c_*. \tag{112}$$

*with some constant $0 < \nu < \min\{\varepsilon_1(c_1 - c_3), d'(u_+) - b'(u_+)\}$.*

**Proof**. From (84) and (6), as set in (88) $V(t,\xi) := U^+(t,x) - \phi(x_1+ct)$ with $\xi = (\xi_1, \xi_2, \cdots, \xi_n) = (x_1+ct, x_2, \cdots, x_n)$, we have

$$\frac{\partial V}{\partial t} + c\frac{\partial V}{\partial \xi_1} - D\Delta V + d(\phi+V) - d(\phi) = \int_{\mathbf{R}^n} f_\alpha(y)[b(\phi+V) - b(\phi)]dy. \tag{113}$$

Applying Taylor expansion formula and noting $(H_3)$ for $d''(u) \geq 0$ and $b''(u) \leq 0$, we have

$$d(\phi+V) - d(\phi) = d'(\phi)V + d''(\bar{\phi}_1)V^2 \geq d'(\phi)V, \tag{114}$$

$$b(\phi+V) - b(\phi) = b'(\phi)V + b''(\bar{\phi}_2)V^2 \leq b'(\phi)V, \tag{115}$$

where $\bar{\phi}_i$ $(i = 1, 2)$ are some functions between $\phi$ and $\phi + V$. Substituting (114) and (115) into (113), we have, for $c > c_*$,

$$(116) \quad \begin{cases} \dfrac{\partial V}{\partial t} + c\dfrac{\partial V}{\partial \xi_1} - D\Delta V + d'(\phi)V \\ \qquad \leq \displaystyle\int_{\mathbf{R}^n} f_\alpha(y)b'(\phi(\xi_1 - y_1 - c\tau))V(t - \tau, \xi - y - c\tau\mathbf{e}_1)dy, \\ \qquad\qquad\qquad\qquad \text{for } t > 0, \xi \in [x_*, \infty) \times \mathbf{R}^{n-1} \\ V|_{\xi_1 = x_*} \leq C_2(1 + t)^{-\frac{n}{2}} e^{-\varepsilon_1(c_1 - c_3)t}, \quad \text{for } t > 0, (\xi_2, \cdots, \xi_n) \in \mathbf{R}^{n-1} \\ V|_{t=s} = V_0(s, \xi), \quad \text{for } s \in [-\tau, 0], \xi \in [x_*, \infty) \times \mathbf{R}^{n-1}, \end{cases}$$

and for $c = c_*$,

$$(117) \quad \begin{cases} \dfrac{\partial V}{\partial t} + c\dfrac{\partial V}{\partial \xi_1} - D\Delta V + d'(\phi)V \\ \qquad \leq \displaystyle\int_{\mathbf{R}^n} f_\alpha(y)b'(\phi(\xi_1 - y_1 - c\tau))V(t - \tau, \xi - y - c\tau\mathbf{e}_1)dy, \\ \qquad\qquad\qquad\qquad \text{for } t > 0, \xi \in [x_*, \infty) \times \mathbf{R}^{n-1} \\ V|_{\xi_1 = x_*} \leq C_3(1 + t)^{-\frac{n}{2}}, \quad \text{for } t > 0, (\xi_2, \cdots, \xi_n) \in \mathbf{R}^{n-1} \\ V|_{t=s} = V_0(s, \xi), \quad \text{for } s \in [-\tau, 0], \xi \in [x_*, \infty) \times \mathbf{R}^{n-1}, \end{cases}$$

provided some positive constants $C_2$ and $C_3$. Here, $x_*$ is selected to be sufficiently large such that, for $\xi_1 \geq x_* \gg 1$,

$$(118) \quad d'(\phi(\xi_1)) \geq \int_{\mathbf{R}^n} f_\alpha(y)b'(\phi(\xi_1 - y_1 - c\tau))dy,$$

due to that, $(H_2)$ for $d'(u_+) > b'(u_+)$ implies

$$\begin{aligned} \lim_{\xi_1 \to \infty} d'(\phi(\xi_1)) = d'(u_+) \quad &> \quad b'(u_+) \\ &= \quad \int_{\mathbf{R}^n} f_\alpha(y)b'(u_+)dy \\ &= \quad \int_{\mathbf{R}^n} f_\alpha(y)\Big[\lim_{\xi_1 \to \infty} b'(\phi(\xi_1 - y_1 - c\tau))\Big]dy \\ &= \quad \lim_{\xi_1 \to \infty} \int_{\mathbf{R}^n} f_\alpha(y)b'(\phi(\xi_1 - y_1 - c\tau))dy, \end{aligned}$$

which further confirms (118) for $\xi_1 \geq x_* \gg 1$.

In the case $c > c_*$, let

$$(119) \quad \bar{V}(t, \xi) = C_4(1 + t + \tau)^{-\frac{n}{2}} e^{-\nu t}, \quad t \in [t_*, \infty)$$

where $C_4 > V_0(s, x) \geq 0$ is a selected large constant, and $t_*$ and $x_*$ are chosen sufficiently large, and $\nu > 0$ is set small within $0 < \nu < \min\{\varepsilon_1(c_1 - c_3), d'(u_+) - $

$b'(u_+)\}$, such that

$$-\frac{n}{2(1+t+\tau)} - \nu + d'(\phi(\xi_1))$$

$$-e^{\nu\tau}\frac{(1+t+\tau)^{\frac{n}{2}}}{(1+t)^{\frac{n}{2}}}\int_{\mathbf{R}^n} f_\alpha(y)b'(\phi(\xi_1 - y_1 - cr))dy$$

$$= [d'(u_+) - b'(u_+)] - \frac{n}{2(1+t+\tau)} - \nu$$

$$-[d'(u_+) - d'(\phi(\xi_1))] - \int_{\mathbf{R}^n} f_\alpha(y)[b'(\phi(\xi_1 - y_1 - cr)) - b'(u_+)]dy$$

$$-\Big[e^{\nu\tau}\Big(\frac{1+t+\tau}{1+t}\Big)^{\frac{n}{2}} - 1\Big]\int_{\mathbf{R}^n} f_\alpha(y)b'(\phi(\xi_1 - y_1 - cr))dy$$

$$(120) \qquad \geq 0, \qquad \text{for } t \geq t_*, \ \xi_1 \geq x_*.$$

We choose $C_4 > 0$ large enough such that $\bar{V}(t,\xi) \geq V(t,\xi)$ for $t \in [0, t_*] \times \mathbf{R}^n$. By a simple but tedious computation as we did in (41), we can verify that $\bar{V}(t,\xi)$ is an upper solution to (116) in the form

$$(121) \quad \begin{cases} \dfrac{\partial \bar{V}}{\partial t} + c\dfrac{\partial \bar{V}}{\partial \xi_1} - D\Delta\bar{V} + d'(\phi)\bar{V} \\ \qquad \geq \displaystyle\int_{\mathbf{R}^n} f_\alpha(y)b'(\phi(\xi_1 - y_1 - c\tau))\bar{V}(t - \tau, \xi - y - c\tau\mathbf{e}_1)dy, \\ \qquad\qquad\qquad\qquad \text{for } t > t_*, \xi \in [x_*, \infty) \times \mathbf{R}^{n-1} \\ \bar{V}|_{\xi_1 = x_*} \geq C_2(1+t)^{-\frac{n}{2}}e^{-\varepsilon_1(c_1-c_3)t}, \quad \text{for } t > t_*, (\xi_2, \cdots, \xi_n) \in \mathbf{R}^{n-1} \\ \bar{V}|_{t=s} \geq V_0(s,\xi), \quad \text{for } s \in [-\tau, 0], \xi \in [x_*, \infty) \times \mathbf{R}^{n-1}. \end{cases}$$

Thus, we get, in the case of $c > c_*$,

$$(122) \quad 0 \leq V(t,\xi) \leq \bar{V}(t,\xi) = C_4(1+\tau+t)^{-\frac{n}{2}}e^{-\nu t}, \quad \text{for } t > t_*, \xi \in [x_*,\infty) \times \mathbf{R}^{n-1}.$$

This proves (111).

When $c = c_*$, we can similarly check that

$$(123) \qquad\qquad \bar{V}(t,\xi) = C_5(1+\tau+t)^{-\frac{n}{2}}, \quad t \in [t_*, \infty)$$

is an upper solution to (116) in the form

$$(124) \quad \begin{cases} \dfrac{\partial \bar{V}}{\partial t} + c\dfrac{\partial \bar{V}}{\partial \xi_1} - D\Delta\bar{V} + d'(\phi)\bar{V} \\ \qquad \geq \displaystyle\int_{\mathbf{R}^n} f_\alpha(y)b'(\phi(\xi_1 - y_1 - c\tau))\bar{V}(t - \tau, \xi - y - c\tau\mathbf{e}_1)dy, \\ \qquad\qquad\qquad\qquad \text{for } t > t_*, \xi \in [x_*, \infty) \times \mathbf{R}^{n-1} \\ \bar{V}|_{\xi_1 = x_*} \geq C_2(1+t)^{-\frac{n}{2}}, \quad \text{for } t > t_*, (\xi_2, \cdots, \xi_n) \in \mathbf{R}^{n-1} \\ \bar{V}|_{t=s} \geq V_0(s,\xi), \quad \text{for } s \in [-\tau, 0], \xi \in [x_*, \infty) \times \mathbf{R}^{n-1}. \end{cases}$$

Then, we get, in the case of $c > c_*$,

$$(125) \qquad 0 \leq V(t,\xi) \leq \bar{V}(t,\xi) \leq C(1+t)^{-\frac{n}{2}}, \quad \text{for } t > 0, \xi \in [x_*,\infty) \times \mathbf{R}^{n-1}.$$

This proves (112). So, the proof of this lemma is complete.

Combing Lemma 3.1 and Lemma 3.2, we obtain the decay rates for $V(t,\xi)$ in $L^\infty(\mathbf{R}^n)$.

**Lemma 3.3.** *It holds that:*

(i) *when $c > c_*$, then*

$$(126) \qquad \|V(t)\|_{L^\infty(\mathbf{R}^n)} \le Ct^{-\frac{n}{2}} e^{-\nu t}, \quad t > 0$$

*with $0 < \nu < \min\{\varepsilon_1(c_1 - c_3), d'(u_+) - b'(u_+)\}$;*

(ii) *when $c = c_*$,*

$$(127) \qquad \|V(t)\|_{L^\infty(\mathbf{R}^n)} \le Ct^{-\frac{n}{2}}, \quad t > 0.$$

Since $V(t, \xi) = U^+(t, x) - \phi(x_1 + ct)$, Lemma 3.3 gives directly the following convergence for the solution in the cases with or without time-delay.

**Lemma 3.4.** *It holds that:*

(i) *when $c > c_*$, then*

$$(128) \qquad \sup_{x \in \mathbf{R}^n} |U^+(t, x) - \phi(x_1 + ct)| \le Ct^{-\frac{n}{2}} e^{-\nu t}, \quad t > 0$$

*with $0 < \nu \le \min\{\varepsilon_1(c_1 - c_3), d'(u_+) - b'(u_+)\}$*

(ii) *when $c = c_*$, then*

$$(129) \qquad \sup_{x \in \mathbf{R}^n} |U^+(t, x) - \phi(x_1 + c_* t)| \le Ct^{-\frac{n}{2}}, \quad t > 0.$$

**Step 2. The convergence of $U^-(t, x)$ to $\phi(x_1 + ct)$**

For any given $c \ge c_*$, let $\xi_1 = x_1 + ct$, $\xi = x + ct\mathbf{e}_1$

$$(130) \qquad v(t, \xi) = \phi(x_1 + ct) - U^-(t, x), \quad v_0(s, \xi) = \phi(x_1 + cs) - \mathcal{U}_0^-(s, x).$$

As in Step 1, we can similarly prove that $U^-(t, x)$ converges to $\phi(x_1 + ct)$ as follows.

**Lemma 3.5.** *It holds that:*

(i) *when $c > c_*$, then*

$$(131) \qquad \sup_{x \in \mathbf{R}^n} |U^-(t, x) - \phi(x_1 + ct)| \le Ct^{-\frac{n}{2}} e^{-\nu t}, \quad t > 0$$

*with $0 < \nu < \min\{\varepsilon_1(c_1 - c_3), d'(u_+) - b'(u_+)\}$;*

(ii) *when $c = c_*$, then*

$$(132) \qquad \sup_{x \in \mathbf{R}^n} |U^-(t, x) - \phi(x_1 + c_* t)| \le Ct^{-\frac{n}{2}}, \quad t > 0.$$

**Step 3. The convergence of $u(t, x)$ to $\phi(x_1 + ct)$**

Finally, we prove that $u(t, x)$ converges to $\phi(x_1 + ct)$. Since the initial data satisfy $U_0^-(s, x) \le u_0(s, x) \le U_0^+(s, x)$ for $(s, x) \in [-\tau, 0] \times \mathbf{R}^n$, where $\tau$ can be taken as $\tau > 0$ or $\tau = 0$, then the comparison principle implies that

$$U^-(t, x) \le u(t, x) \le U^+(t, x), \quad (t, x) \in R_+ \times \mathbf{R}^n.$$

Thanks to Lemmas 3.4 and 3.5, by the squeeze argument, we have the following convergence results.

**Lemma 3.6.** *It holds that:*

(i) *when $c > c_*$, then*

$$(133) \qquad \sup_{x \in \mathbf{R}^n} |u(t, x) - \phi(x_1 + ct)| \le Ct^{-\frac{n}{2}} e^{-\nu t}, \quad t > 0$$

*with $0 < \nu < \min\{\varepsilon_1(c_1 - c_3), d'(u_+) - b'(u_+)\}$;*

(ii) *when $c = c_*$, then*

$$(134) \qquad \sup_{x \in \mathbf{R}^n} |u(t, x) - \phi(x_1 + c_* t)| \le Ct^{-\frac{n}{2}}, \quad t > 0.$$

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