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STABILITY OF SHOCK PROFILES FOR NONCONVEX SCALAR VISCOUS CONSERVATION LAWS

MING MEI
Department of Mathematics, Faculty of Science, Kanazawa University, Kakuma-Machi, Kanazawa 920-11, Japan

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This paper is to study the stability of shock profiles for nonconvex scalar viscous conservation laws with the nondegenerate and the degenerate shock conditions by means of an elementary energy method. In both cases, the shock profiles are proved to be asymptotically stable for suitably small initial disturbances. Moreover, in the case of nondegenerate shock condition, time decay rates of asymptotics are also obtained.

1. Introduction

In this paper we investigate the asymptotic stability of shock profiles for nonconvex scalar viscous conservation laws of the form

$$u_t + f(u)_x = \mu u_{xx}, \quad x \in \mathbb{R}^1, \quad t > 0\,, \tag{1.1}$$

with the initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^1\,, \tag{1.2}$$

where $\mu > 0$ is a constant, $f(u)$ is a smooth function satisfying

$$f''(u) \lesseqgtr 0 \quad \text{for} \quad u \lesseqgtr 0 \quad \text{under consideration}\,, \tag{1.3}$$

and the initial data satisfy

$$\lim_{x \to \pm\infty} u_0(x) = u_\pm \tag{1.4}$$

for some given constants $u_\pm$. Let $u_+ \neq u_-$ and let s be a constant. If $u = U(x-st)$ is a smooth solution of (1.1) satisfying $U(\pm\infty) = u_\pm$, then we call $U(x-st)$ a shock profile of (1.1) which connects u_+ and u_-, and call s a shock speed. Note that $f(u)$ has a point of inflection at $u = 0$.

The stability problem of the shock profiles $U(x - st)$ have a long history beginning from the paper of Il'in–Oleinik.[2] Based on the maximum principle, they showed the asymptotic stability and a time decay rate of convergence when $f(u)$ is strictly convex, i.e. $f''(u) > 0$. A new different approach based on an energy method which can also be applied to systems was introduced independently by Matsumura–Nishihara[7] and by Goodman.[1] Since then, the stability problem has been studied by many mathematicians (see Refs. 3–6, 8 and 9). Among them, Kawashima-Matsumura[4] proved the asymptotic stability and an algebraic time decay rate of asymptotics like $t^{-\gamma}$ $(\gamma > 0)$ of shock profiles for problem (1.1), (1.2) with the convex nonlinearity by an energy method in some weighted Sobolev spaces. After this paper, by applying a similar energy method which makes use of a weight function depending on the shock profiles, they obtained in Ref. 5 a stability result also in the nonconvex case (1.3) with the nondegenerate shock condition $[f'(u_+) < s < f'(u_-)]$, but they did not show any time decay rate of the asymptotics. On the other hand, applying the spectral analysis, Jones–Gardner–Kapitula[3] proved the stability and a time decay rate for a general class of nonconvex scalar viscous conservation laws with the nondegenerate shock condition. However, their time decay rate is less sufficient than that of Kawashima-Matsumura[4] in the case where $f''(u) > 0$. Here we should emphasize that all the previous works do not cover the case of degenerate shock condition $[f'(u_+) = s < f'(u_-)]$.

In this paper we discuss the stability of the shock profiles in the case of the nonconvex nonlinearity (1.3). Our main goal is to show the stability under the degenerate shock condition and also to obtain time decay rates of the asymptotics under the nondegenerate shock condition. In Sec. 2, we give a statement on the existence and uniqueness of shock profiles together with their properties. In Sec. 3, we discuss the stability of the shock profiles and prove time decay rates under the nondegenerate shock condition. When the integral of the initial disturbance over $(-\infty, x]$, say $\phi_0(x)$, has a polynomial decay order $O(|x|^{-\alpha/2})$ in the sense that $|x|^{\alpha/2}\phi_0 \in L^2$ for some $\alpha > 0$, we obtain a time decay rate $t^{-\gamma}$ $(\gamma = [\alpha]/2)$. This time decay rate is better than that in Jones–Gardner–Kapitula,[3] and seems to be almost optimal from a view point of the optimality shown by Nishihara[8] for $f = u^2/2$. Moreover, we obtain a better time decay rate like $e^{-\theta t}$ $(\theta > 0)$ for a class of $\phi_0(x)$ which has an exponential decay order. This exponential time decay corresponds to the one shown by Il'in–Oleinik[2] in the case of convex nonlinearity. In Sec. 4, under the degenerate shock condition, we prove the stability of the shock profiles by introducing a suitable weight function and changing the unknown function. Finally in Sec. 5, we make a short remark on the stability results when $f(u)$ verifies the opposite sign condition to (1.3).

Notations. L^2 denotes the space of measurable functions on $\mathbb{R}$ which are square integrable, with the norm

$$\|f\| = \left(\int |f(x)|^2 dx\right)^{1/2}.$$

H^l $(l \geq 0)$ denotes the Sobolev space of L^2-functions f on $\mathbb{R}$ whose derivatives $\partial_x^j f$, $j = 1, \dots, l$, are also L^2-functions, with the norm

$$\|f\|_l = \left(\sum_{j=0}^{l} \|\partial_x^j f\|^2 \right)^{1/2} .$$

L^2_α $(\alpha \in \mathbb{R})$ denotes the space of measurable functions on $\mathbb{R}$ which satisfy $\langle x \rangle^{\alpha/2} f \in L^2$, with the norm

$$|f|_\alpha = \left(\int \langle x \rangle^\alpha |f(x)|^2 dx \right)^{1/2} ,$$

where $\langle x \rangle = (1 + x^2)^{1/2}$. Note that $L^2 = H^0 = L^2_0$ and $\|\cdot\| = \|\cdot\|_0 = |\cdot|_0$.

Let T and B be a positive constant and a Banach space, respectively. $C^k(0, T; B)$ $(k \geq 0)$ denotes the space of B-valued k times continuously differentiable functions on $[0, T]$, and $L^2(0, T; B)$ denotes the space of B-valued L^2-functions on $[0, T]$. The corresponding spaces of B-valued function on $[0, \infty)$ are defined similarly.

2. Properties of Shock Profiles

Under the assumption (1.3), Kawashima–Matsumura[5] proved that there exists a shock profile $U(x - st)$ connecting u_+ and u_- if and only if $u_\pm$ and s satisfy the Rankine–Hugoniot condition

$$-s(u_+ - u_-) + (f(u_+) - f(u_-)) = 0 \tag{2.1}$$

and the generalized shock condition

$$f'(u_+) \leq s < f'(u_-) . \tag{2.2}$$

It includes the following two cases: the nondegenerate shock condition

$$f'(u_+) < s < f'(u_-) \tag*{$(2.2)_1$}$$

and the degenerate shock condition

$$f'(u_+) = s < f'(u_-) . \tag*{$(2.2)_2$}$$

We also call the conditions $(2.2)_1$ and $(2.2)_2$ Lax's shock condition and Oleinik's shock condition respectively. It is easy to see that under the conditions (2.1) and (2.2), a shock profile $U(\xi)$ $(\xi = x - st)$ must satisfy the following ordinary differential equation

$$\mu U_\xi = -sU + f(U) - a \equiv h(U) , \tag{2.3}$$

where a is an integral constant defined by $a = -su_\pm + f(u_\pm)$. To simplify the situation, we may assume without loss of generality

$$u_- < 0 < u_+ , \tag{2.4}$$

in the rest of this paper. Now we are ready to summarize a characterization of the generalized shock condition (2.2) and the results on the existence of shock profiles studied in Ref. 5:

Proposition 2.1.[5] *Suppose that* (1.3) *and* (2.4) *hold. Then the following statements are equivalent:*

(i) *The generalized shock condition* (2.2) *holds.*

(ii) $h(u) > 0$ *for* $u \in (u_-, u_+)$.

(iii) *There exists a unique* $u_* \in (u_-, u_+)$ *such that* $f'(u_*) = s$ *and the following holds*

$$f'(u) < s \quad for \quad u \in (u_*, u_+), \quad s < f'(u) \quad for \quad u \in (u_-, u_*). \tag{2.5}$$

Moreover, the u_* *in* (iii) *verifies* $u_* < 0$.

Proposition 2.2.[5] *Suppose that* (1.3) *and* (2.4) *hold.*

(i) *If* (1.1) *admits a shock profile* $U(x - st)$ *connecting* u_- *and* u_+, *then* $u_\pm$ *and* s *must satisfy the Rankine–Hugoniot condition* (2.1) *and the generalized shock condition* (2.2).

(ii) *Conversely, suppose that* (2.1) *and* (2.2) *hold, then there exists a shock profile* $U(x - st)$ *of* (1.1) *which connects* $u_\pm$. *The* $U(\xi)$ $(\xi = x - st)$ *is unique up to a shift in* ξ *and is a monotone function of* ξ. *In particular, we have*

$$u_- < U(\xi) < u_+, \quad U_\xi(\xi) > 0 \tag{2.6}$$

for all $\xi \in \mathbb{R}$. *Moreover,* $U(\xi) \to u_\pm$ *exponentially as* $\xi \to \pm\infty$, *with the following exceptional case: when* $f'(u_+) = s$, $U(\xi) \to u_+$ *at the rate* $|\xi|^{-1}$ *as* $\xi \to +\infty$.

In later sections, we often make use of a function $Z(u)$ defined by

$$Z(u) = \frac{1}{4\mu}(f'(u) - s)^2 - \frac{1}{2\mu} f''(u)h(u), \tag{2.7}$$

whose basic properties are given by

Lemma 2.3. (Properties of $Z(u)$) *Suppose that* (1.3), (2.2) *and* (2.4) *hold. Moreover, suppose that*

$$f'''(u) > 0 \quad for \quad u \in [u_-, u_+], \quad u \neq 0. \tag{2.8}$$

Then the function $Z(u)$ *satisfies the following properties:*

(i) Z *is a positive and decreasing function in* (u_-, u_+).

(ii) $Z(u) \geq \frac{1}{4\mu}(f'(u_+) - s)^2 \qquad for \ u \in [u_-, u_+]$.

(iii) *In the case* $(2.2)_2$, *it holds that* $|h(u)| = O(|u - u_+|^2)$ *and* $Z(u) = O(|u - u_+|^3)$ *in a neighborhood of* $u = u_+$.

(iv) *There exists a constant* C_0 *such that*

$$(f'(u) - s)^2 |h(u)|^{1/2} \leq C_0 Z(u) \quad for \quad u \in [u_-, u_+].$$

Proof. All these assertions are easily proved by elementary calculations. In fact, if we differentiate $Z(u)$ with respect to u, we have

$$Z'(u) = \frac{-1}{2\mu} f'''(u) h(u)$$

which implies, by Proposition 2.1 and (2.8), that $Z'(u) < 0$ for all $u \in (u_-, u_+)$, $u \neq 0$. Hence, noting that $h(u_\pm) = 0$, we easily get (i) and (ii). For the proof of (iii) and (iv), we may use Taylor's formula around $u = u_+$. We omit the details. □

3. Time Decay Estimates (Nondegenerate Case)

In this section, we assume the Rankine–Hugoniot condition (2.1) and the non-degenerate shock condition $(2.2)_1$. Let $U(x - st)$ be a shock profile connecting $u_\pm$, and let us define x_0 by

$$\int_{-\infty}^{+\infty} (u_0(x) - U(x))dx = x_0(u_+ - u_-)\,. \tag{3.1}$$

We note that x_0 is uniquely determined by (3.1), provided that $u_0 - U$ is integrable over $\mathbb{R}$. Then the shifted function $U(x - st + x_0)$ is also a shock profile connecting $u_\pm$ such that

$$\int_{-\infty}^{+\infty} (u_0(x) - U(x + x_0))dx = 0\,. \tag{3.2}$$

In Kawashima–Matsumura,[5] this shifted shock profile $U(x - st + x_0)$ was proved to be stable as $t \to \infty$, provided the following function $\phi_0(x)$ is suitably small:

$$\phi_0(x) = \int_{-\infty}^{x} (u_0(y) - U(y + x_0))dy\,. \tag{3.3$_1$}$$

In this paper, we show not only the stability but also its time decay of convergence with the same rate as in the case of convex nonlinearity studied in Ref. 4.

Theorem 3.1. *Suppose that* (1.3), (2.1), $(2.2)_1$, (2.4) *and* (2.8) *hold. Let* $U(x - st)$ *be a shock profile connecting* $u_\pm$, *and suppose* $u_0(x) - U(x)$ *is integrable over* $\mathbb{R}$. *We define* x_0 *and* $\phi_0(x)$ *by* (3.1) *and* $(3.3)_1$. *Suppose that* $\phi_0 \in H^2 \cap L^2_\alpha$ *for some* $\alpha \geq 0$. *Then there exists a positive constant* δ_0 *such that if* $\|u_0 - U\|_1 + |\phi_0|_\alpha \leq \delta_0$, *the Cauchy problem* (1.1), (1.2) *has a unique global solution* $u(t, x)$ *satisfying*

$$u - U \in C^0(0, \infty; H^1) \cap L^2(0, \infty; H^2)\,,$$

where $U = U(x - st + x_0)$ *is the shifted shock profile. Moreover, the solution verifies the following decay rate estimate*

$$\sup_{x \in \mathbb{R}} |u(t, x) - U(x - st + x_0)| \leq C(1 + t)^{-\gamma/2}(\|u_0 - U\|_1 + |\phi_0|_\alpha)\,, \tag{3.4}$$

where $\gamma = [\alpha]$.

The decay rate $t^{-[\alpha]/2}$ in Theorem 3.1 is better than that in Jones–Gardner–Kapitula[3] because they showed the rate $t^{-[\alpha/2]/2}$ for the corresponding α. For positive integers α, this rate also seems optimal in L^2-framework from the arguments in Nishihara.[8] Furthermore, we can show the asymptotic stability with an exponential decay rate $e^{-\theta t/2}$ (with some positive constant θ) for a class of initial data such that the following function $\Phi_0(x)$ is suitably small:

$$\Phi_0(x) = \sqrt{\mu}\phi_0(x - x_0)h(U(x))^{-1/2}\,. \tag{3.3$_2$}$$

This exponential time decay rate is somehow corresponding to that in Il'in–Oleinik[2] in the case of convex nonlinearity.

Theorem 3.2. *Suppose that* (1.3), (2.1), $(2.2)_1$, (2.4) *and* (2.8) *hold. Let* $U(x-st)$ *be a shock profile connecting* $u_\pm$, *and suppose that* $u_0(x) - U(x)$ *is integrable over* $\mathbb{R}$ *and that* $\Phi_0(x)$ *is well defined by* $(3.3)_2$ *and is in* H^2. *Then there exists a positive constant* δ_1 *such that if* $\|\Phi_0\|_2 \le \delta_1$, *the Cauchy problem* (1.1), (1.2) *has a unique global solution* $u(t,x)$ *satisfying*

$$u - U \in C^0(0,\infty; H^1) \cap L^2(0,\infty; H^2)\,,$$

where $U = U(x - st + x_0)$ *is the shifted shock profile. Moreover, the solution verifies the decay rate estimate*

$$\sup_{x\in\mathbb{R}} |u(t,x) - U(x - st + x_0)| \le Ce^{-\theta t/2}\|\Phi_0\|_2\,, \tag{3.5}$$

where θ *is any fixed constant satisfying* $0 < \theta \leqslant \frac{1}{2\mu}(f'(u_+) - s)^2$.

Remark. By using Proposition 2.2, we can see that the weight function $h(U(x))^{-1/2}$ in $(3.3)_2$ has an exponential decay order $O(e^{-C_\pm|x|})$ ($C_\pm = |s - f'(u_\pm)|/2\mu > 0$) as $x \to \pm\infty$. Therefore, the condition $\|\Phi_0\|_2 \le \delta_1$ implies that the initial function $\phi_0(x)$ must have at least the same exponential decay order as the weight function has.

3.1. *Proof of Theorem 3.1*

The proof is given by combining the arguments in Refs. 4 and 5. As in the previous papers, we seek solutions of (1.1), (1.2) in the form

$$u(t,x) = U(\xi) + \phi_\xi(t,\xi), \quad \xi = x - st + x_0\,. \tag{3.6}$$

Then the problem (1.1), (1.2) is reduced to

$$\phi_t - (s - f'(U))\phi_\xi - \mu\phi_{\xi\xi} = F(U,\phi_\xi)\,, \tag{3.7}$$

$$\phi(0,\xi) = \phi_0(\xi - x_0)\,, \tag{3.8}$$

where

$$F = -\{f(U + \phi_\xi) - f(U) - f'(U)\phi_\xi\}\,. \tag{3.9}$$

The problem (3.7), (3.8) can be solved globally in time as follows.

Theorem 3.3. *Suppose that $\phi_0 \in H^2 \cap L^2_\alpha$ for some $\alpha \geq 0$, and the conditions in Theorem* 3.1 *hold. Then there exists a positive constant δ_2 such that if $\|\phi_0\|_2 \leq \delta_2$, the problem* (3.7), (3.8) *has a unique global solution $\phi(t,\xi)$ satisfying*

$$\phi \in C^0(0,\infty;H^2) \cap C^1(0,\infty;L^2), \quad \phi_\xi \in L^2(0,\infty;H^2)\,, \tag{3.10}$$

and the decay estimate

$$(1+t)^{[\alpha]}\|\phi(t)\|_2^2 + \int_0^t (1+\tau)^{[\alpha]}\|\phi_\xi(\tau)\|_2^2 d\tau \leq C(|\phi_0|_\alpha^2 + \|\phi_{0,\xi}\|_1^2) \tag{3.11}$$

holds for $t \geq 0$.

Since we can easily prove Theorem 3.1 from Theorem 3.3, it is sufficient to prove Theorem 3.3 for our purpose. To do that, we shall combine a local existence result together with *a priori* estimates.

Proposition 3.4. (local existence) *Suppose that $\phi_0 \in H^2$ and the conditions in Theorem* 3.1 *hold. Then there is a positive constant T_0 such that the problem* (3.7), (3.8) *has a unique solution $\phi(t,\xi)$ satisfying*

$$\begin{gathered}\phi \in C^0(0,T_0;H^2) \cap C^1(0,T_0;L^2), \quad \phi_\xi \in L^2(0,T_0;H^2)\,, \\ \sup_{t\in[0,T_0]} \|\phi(t)\|_2 \leq 2\|\phi_0\|_2\,.\end{gathered} \tag{3.12}$$

Moreover, if $\phi_0 \in L^2_\alpha$ for some $\alpha \geq 0$, then $\phi \in C^0(0,T_0;L^2_\alpha)$ and $\phi_\xi \in L^2(0,T_0;L^2_\alpha)$.

Proposition 3.5. (*a priori* estimate) *Let T be a positive constant, and $\phi(t,\xi)$ be a solution of the problem* (3.7), (3.8) *satisfying*

$$\phi \in C^0(0,T;H^2 \cap L^2_\alpha) \cap C^1(0,T;L^2), \quad \phi_\xi \in L^2(0,T;H^2 \cap L^2_\alpha) \tag{3.13}$$

for some $\alpha \geq 0$. Then there exist positive constants δ_3 and C independent of T such that if $\sup_{0\leq t\leq T} \|\phi(t)\|_2 \leq \delta_3$, then the estimate

$$(1+t)^{[\alpha]}\|\phi(t)\|_2^2 + \int_0^t (1+\tau)^{[\alpha]}\|\phi_\xi(\tau)\|_2^2 d\tau \leq C(|\phi_0|_\alpha^2 + \|\phi_{0,\xi}\|_1^2) \tag{3.14}$$

holds for $t \in [0,T]$.

Since Proposition 3.4 can be proved in the standard way, we omit its proof. Once Proposition 3.5 is proved, using the continuation arguments based on Propositions 3.4 and 3.5, we can show Theorem 3.3. This scheme is the same as in Refs. 4 and 5, so we also omit its proof. To prove Proposition 3.5, we prepare the following several lemmas. Now, setting

$$N(t) = \sup_{0\leq\tau\leq t} \|\phi(t)\|_2, \quad (t \in [0,T])\,, \tag{3.15}$$

$$N_\alpha = |\phi_0|_\alpha + \|\phi_{0,\xi}\|_1\,, \tag{3.16}$$

we first have

Key Lemma 3.6. *For any $\beta, \gamma \in [0, \alpha]$, there exists a positive constant C independent of T, β and γ such that the estimate*

$$\begin{aligned}(1+t)^\gamma|\phi(t)|_\beta^2 &+ \beta\int_0^t(1+\tau)^\gamma|\phi(\tau)|_{\beta-1}^2d\tau + \int_0^t(1+\tau)^\gamma|\phi_\xi(\tau)|_\beta^2d\tau \\ &\leq C\left\{|\phi_0|_\beta^2 + \gamma\int_0^t(1+\tau)^{\gamma-1}|\phi(\tau)|_\beta^2 + \beta\int_0^t(1+\tau)^\gamma\|\phi_\xi(\tau)\|^2d\tau\right. \\ &\quad\left. + \int_0^t\int(1+\tau)^\gamma\langle\xi\rangle^\beta|\phi||F(U,\phi_\xi)|d\xi d\tau\right\} \qquad (3.17)\end{aligned}$$

holds for $t \in [0, T]$.

Proof. Thanks to Propositions 2.1 and 2.2, there exists a unique number ξ_* such that $u_* = U(\xi_*)$ and $u_- < U(\xi_*) < 0$. Multiplying (3.7) by $(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(U)\phi$, where $w(u)$ is a weight function which will be chosen such that $w(u) \in C^1[u_-, u_+]$ and $w(u) > 0$, we have

$$\begin{aligned}&\left\{\frac{1}{2}(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(U)\phi^2\right\}_t - \frac{\gamma}{2}(1+t)^{\gamma-1}\langle\xi-\xi_*\rangle^\beta w(U)\phi^2 \\ &\quad - \left\{\frac{1}{2}(1+t)^\gamma(s-f'(U))\langle\xi-\xi_*\rangle^\beta w(U)\phi^2\right\}_\xi \\ &\quad - \{\mu(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(U)\phi\phi_\xi\}_\xi \\ &\quad + \frac{1}{2}(1+t)^\gamma\langle\xi-\xi_*\rangle^{\beta-1}w(U)\{\beta(s-f'(U))\langle\xi-\xi_*\rangle^{-1}(\xi-\xi_*) \\ &\quad - f'(U)_\xi\langle\xi-\xi_*\rangle + \frac{w'(U)}{w(U)}U_\xi\langle\xi-\xi_*\rangle(s-f'(U))\}\phi^2 \\ &\quad + \mu(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(U)\phi_\xi^2 + \mu w(U)\{\beta(1+t)^\gamma\langle\xi-\xi_*\rangle^{\beta-2}(\xi-\xi_*) \\ &\quad + \frac{w'(U)}{w(U)}U_\xi(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\}\phi\phi_\xi \\ &= (1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(U)F\phi\,. \qquad (3.18)\end{aligned}$$

Since the Schwarz inequality gives

$$\begin{aligned}&\left|\mu\frac{w(U)_\xi}{w(U)}(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\phi\phi_\xi\right| \\ &\qquad \leq \varepsilon\mu(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\phi_\xi^2 + \frac{\mu}{4\varepsilon}(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta\left[\frac{w'(U)}{w(U)}U_\xi\right]^2\phi^2 \qquad (3.19)\end{aligned}$$

for any $\varepsilon > 0$, after substituting (3.19) into (3.18), we have

$$\begin{aligned}
&\left\{\frac{1}{2}(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(U)\phi^2\right\}_t - \frac{\gamma}{2}(1+t)^{\gamma-1}\langle\xi-\xi_*\rangle^\beta w(U)\phi^2 \\
&\quad - \left\{\frac{1}{2}(1+t)^\gamma(s-f'(U))\langle\xi-\xi_*\rangle^\beta w(U)\phi^2\right\}_\xi \\
&\quad - \{\mu(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(U)\phi\phi_\xi\}_\xi + (1+t)^\gamma\langle\xi-\xi_*\rangle^{\beta-1}w(U)A_\beta(\xi)\phi^2 \\
&\quad + (1-\varepsilon)\mu(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(U)\phi_\xi^2 \\
&\quad + \mu w(U)\beta(1+t)^\gamma\langle\xi-\xi_*\rangle^{\beta-2}(\xi-\xi_*)\phi\phi_\xi \\
&\le (1+t)^\gamma\langle\xi-\xi_*\rangle^\beta w(U)F\phi\,,
\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
A_\beta(\xi) = \frac{1}{2}\bigg\{&\beta(s-f'(U))\langle\xi-\xi_*\rangle^{-1}(\xi-\xi_*) - f'(U)_\xi\langle\xi-\xi_*\rangle \\
&+ \frac{w'(U)}{w(U)}U_\xi\langle\xi-\xi_*\rangle(s-f'(U)) - \frac{\mu}{2\varepsilon}\left[\frac{w'(U)}{w(U)}U_\xi\right]^2\langle\xi-\xi_*\rangle\bigg\}\,.
\end{aligned} \tag{3.21}$$

Now we are going to look for a suitable weight function $w(U)$ such that for a suitable $\varepsilon > 0$,

$$A_\beta(\xi) \ge \beta C_1, \quad (C_1 > 0 \text{ is independent of } \beta)\,. \tag{3.22}$$

Inspired by the arguments in Ref. 5, we choose this weight function $w(U)$ as

$$w(u) = \begin{cases} 1, & u \in [u_-, 0] \\ C_2(s-f'(u))^{-2}, & u \in [0, u_+] \end{cases}$$

where $C_2 = (s-f'(0))^2$. We note that $w(u)$ is a C^1-function on $[u_-, u_+]$. To verify (3.22), we define ξ_{**} by $U(\xi_{**}) = 0$. This ξ_{**} is uniquely determined and satisfies $\xi_* < \xi_{**}$. We divide into two cases: $\xi \le \xi_{**}$ and $\xi \ge \xi_{**}$.

Case 1. $\xi \le \xi_{**}$. In this case, we have $u_- < U \le 0$ so that $w(U) = 1$ and $f''(U) \le 0$. The $A_\beta(\xi)$ in (3.21) becomes

$$A_\beta(\xi) = \frac{1}{2}\{\beta(s-f'(U))\langle\xi-\xi_*\rangle^{-1}(\xi-\xi_*) - \mu^{-1}f''(U)h(U)\langle\xi-\xi_*\rangle\}\,. \tag{3.23}$$

Let $g(\xi) = s - f'(U(\xi))$. We see that $g(\xi)$ is strictly increasing on $(-\infty, \xi_{**}]$ and verifies $g(\xi_*) = 0$, $g'(\xi_*) = -\mu^{-1}f''(u_*)h(u_*) > 0$ and $g(-\infty) = s - f'(u_-) < 0$. Therefore, it easily holds that

$$(\xi-\xi_*)\langle\xi-\xi_*\rangle^{-1}(s-f'(U)) \ge \begin{cases} \frac{1}{2}g'(\xi_*)(\xi-\xi_*)^2, & \text{for } \xi \text{ near } \xi_*\,, \\ C, & \text{otherwise}\,, \end{cases}$$

where C is a positive constant. On the other hand, we have

$$-\mu^{-1}\langle\xi-\xi_*\rangle f''(U)h(U)=\langle\xi-\xi_*\rangle g'(\xi)\ge 0$$

for $\xi\in(-\infty,\xi_{**}]$, and in particular,

$$-\mu^{-1}\langle\xi-\xi_*\rangle f''(U)h(U)\ge g'(\xi_*)/2$$

for ξ near ξ_*. These observation prove (3.23) for $\xi\le\xi_{**}$.

Case 2. $\xi\ge\xi_{**}$. In this case, we have $0\le U<u_+$ so that $w(U)=C_2(s-f'(U))^{-2}$. Therefore, the $A_\beta(\xi)$ in (3.21) is reduced to

$$\begin{aligned}A_\beta(\xi)=\frac{1}{2}\{&\beta(s-f'(U))\langle\xi-\xi_*\rangle^{-1}(\xi-\xi_*)\\&+(1-r(U)/\varepsilon)\mu^{-1}f''(U)h(U)\langle\xi-\xi_*\rangle\}\,,\end{aligned}\tag{3.24}$$

where $r(u)=2h(u)(s-f'(u))^{-2}f''(u)$. Since $f'''(u)>0$, $h(u_+)=0$ and $f''(0)=0$, we have $r(u_+)=r(0)=0$ and $r(u)>0$ for $u\in(0,u_+)$. Furthermore, by the same arguments in Ref. 5, we can check

$$0<r(u)\le\bar r<1\quad\text{for}\quad u\in(0,u_+)\,,$$

where $\bar r=\max\limits_{u\in[0,u_+]}r(u)$. In fact, $r(u)$ attains its maximum over $[0,u_+]$ at a point $u=\bar u$ in $(0,u_+)$, and hence $\bar r=r(\bar u)>0$ and $r'(\bar u)=0$. By a straightforward computation, we can evaluate $\bar r$ by

$$\bar r=1-h(\bar u)f'''(\bar u)/(s-f'(\bar u))f''(\bar u)\,.$$

Since $\bar u\in(0,u_+)$, using (1.3), (2.8) and Proposition 2.1 in the above equality, we find that $\bar r<1$.

Now we choose $\varepsilon>0$ such that $\bar r<\varepsilon<1$, and put $C\equiv 1-\bar r/\varepsilon>0$, we have from (3.24)

$$A_\beta(\xi)\ge\frac{1}{2}\{\beta(s-f'(U))\langle\xi-\xi_*\rangle^{-1}(\xi-\xi_*)+C\mu^{-1}f''(U)h(U)\langle\xi-\xi_*\rangle\}\,.$$

Using this expression, we can prove (3.22) for $\xi\le\xi_{**}$ by the same arguments as in Case 1. Thus, we have proved (3.22) for all $\xi\in\mathbb{R}$.

Integrating (3.20) over $[0,t]\times\mathbb{R}$ and using the estimate (3.22), we get

$$\begin{aligned}(1+t)^\gamma|\phi(t)|_\beta^2&+\beta\int_0^t(1+\tau)^\gamma|\phi(\tau)|_{\beta-1}^2d\tau+\int_0^t(1+\tau)^\gamma|\phi_\xi(\tau)|_\beta^2d\tau\\&\le C\bigg\{|\phi_0|_\beta^2+\gamma\int_0^t(1+\tau)^{\gamma-1}|\phi(\tau)|_\beta^2d\tau\\&\quad+\beta\int_0^t\int(1+\tau)^\gamma\langle\xi\rangle^{\beta-1}|\phi\phi_\xi|d\xi d\tau\\&\quad+\int_0^t\int(1+\tau)^\gamma\langle\xi\rangle^\beta|\phi||F(U,\phi_\xi)|d\xi d\tau\bigg\}\,.\end{aligned}\tag{3.25}$$

To show the desired estimate (3.17), we must estimate the third term on the right-hand side of (3.25). Using Schwarz' inequality, we have

$$\beta C \int \langle \xi \rangle^{\beta-1} |\phi \phi_\xi| d\xi \leq \frac{\beta}{2} |\phi|_{\beta-1}^2 + \beta C \int \langle \xi \rangle^{\beta-1} \phi_\xi^2 d\xi \,.$$

We choose a number r so large that $\alpha C \langle \xi \rangle^{-1} \leq 1/2$ for any $|\xi| \geq r$, and divide the integral on the right-hand side into two parts I_1 and I_2 according to the regions $|\xi| \geq r$ and $|\xi| \leq r$. Then we have the estimates $I_1 \leq \frac{1}{2}|\phi_\xi|_\beta^2$ and $I_2 \leq \beta C \|\phi_\xi\|^2$ with some constant C. Substitution of these estimates into (3.25) yields (3.17). This completes the proof of Key Lemma 3.6. □

For the derivatives of the solution, we can obtain the following estimates in relatively easier way. We omit the details.

Lemma 3.7. *Let $l = 1$ or 2. For any $\gamma \in [0, \alpha]$, there exists a positive constant C independent of T and γ such that the estimate*

$$\begin{aligned} &(1+t)^\gamma \|\partial_\xi^l \phi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\partial_\xi^{l+1} \phi(\tau)\|^2 d\tau \\ &\quad \leq C \bigg\{ \|\partial_\xi^l \phi_0\|^2 + \int_0^t (1+\tau)^\gamma \|\phi_\xi(\tau)\|_{l-1}^2 d\tau \\ &\qquad + \int_0^t \int (1+\tau)^\gamma |\partial_\xi^{l+1} \phi| |\partial_\xi^{l-1} F| d\xi d\tau \bigg\} \end{aligned} \tag{3.26$_l$}$$

holds for $t \in [0, T]$.

Now, by Key Lemma 3.6, we can prove

Lemma 3.8. *Let $\gamma \in [0, \alpha] \cap \mathbb{Z}$. Then there exist positive constants δ_4 and $C = C(\delta_4)$ independent of T and γ such that if $N(T) \leq \delta_4$, then*

$$(1+t)^\gamma \|\phi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\phi_\xi(\tau)\|^2 d\tau \leq C N_\alpha^2 \tag{3.27}$$

holds for $t \in [0, T]$.

Proof. At first, we should estimate the last integral on the right-hand side of (3.17). By Taylor's formula and Sobolev's embedding theorem, it is majorized by

$$C N(t) \int_0^t (1+\tau)^\gamma |\phi_\xi(\tau)|_\beta^2 d\tau \,,$$

with a positive constant C. Therefore, for small $N(T)$, say $N(T) \leq \delta_4$, the inequality (3.17) becomes suitably

$$\begin{aligned} &(1+t)^\gamma |\phi(t)|_\beta^2 + \beta \int_0^t (1+\tau)^\gamma |\phi(\tau)|_{\beta-1}^2 d\tau + \int_0^t (1+\tau)^\gamma |\phi_\xi(\tau)|_\beta^2 d\tau \\ &\quad \leq C \bigg\{ |\phi_0|_\beta^2 + \gamma \int_0^t (1+\tau)^{\gamma-1} |\phi(\tau)|_\beta^2 d\tau + \beta \int_0^t (1+\tau)^\gamma \|\phi_\xi(\tau)\|^2 d\tau \bigg\} \end{aligned} \tag{3.28}$$

with a positive constant $C = C(\delta_4)$. When $0 \le \alpha < 1$ (so $\gamma = 0$), choose $\beta = 0$ in (3.28), we obtain (3.27), i.e.

$$\|\phi(t)\|^2 + \int_0^t \|\phi_\xi(\tau)\|^2 d\tau \le CN_\alpha^2 . \tag{3.29}$$

When $1 \le \alpha < 2$ (so $\gamma = 0$ or 1), choosing $\gamma = 1$ and $\beta = 0$ in (3.28), we first have

$$(1+t)\|\phi(t)\|^2 + \int_0^t (1+\tau)\|\phi_\xi(\tau)\|^2 d\tau \le C\left\{\|\phi_0\|^2 + \int_0^t \|\phi(\tau)\|^2 d\tau\right\}. \tag{3.30}$$

Secondly, choosing $\gamma = 0$ and $\beta = 1$ in (3.28), we have

$$\int_0^t \|\phi(\tau)\|^2 d\tau \le C\left\{|\phi_0|_1^2 + \int_0^t \|\phi_\xi(\tau)\|^2 d\tau\right\}. \tag{3.31}$$

By (3.29), (3.30) and (3.31), we can prove (3.27) for $1 \le \alpha < 2$. Repeating the same procedure, we can get the desired estimate (3.27) for any $\alpha \ge 0$. □

Combining (3.27) together with $(3.26)_l$, we can prove

Lemma 3.9. *Let $l = 1$ or 2. For any $0 \le \gamma \le [\alpha]$, there exist positive constants $\delta_5(\le \delta_4)$ and $C = C(\delta_5)$ independent of T and γ such that if $N(T) \le \delta_5$, then the estimate*

$$(1+t)^\gamma \|\partial_\xi^l \phi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\partial_\xi^{l+1}\phi(\tau)\|^2 d\tau \le CN_\alpha^2 \tag{3.32$_l$}$$

holds for $t \in [0, T]$.

Proof of Proposition 3.5. Let δ_3 be chosen as $0 < \delta_3 \le \delta_5$. Then combining (3.27) together with $(3.32)_l$, we can directly prove the estimate (3.14). □

3.2. *Proof of Theorem 3.2*

In this case, we seek the solution of (1.1), (1.2) in the form

$$\begin{aligned} u(t,x) &= U(\xi) + \phi_\xi(t,\xi), \\ \phi &= \mu^{-1/2}|h(U(\xi))|^{1/2}\Phi(t,\xi), \quad \xi = x - st + x_0 . \end{aligned} \tag{3.33}$$

Since we assumed that $u_- < 0 < u_+$, we have $h(U) > 0$ so that $\phi = \mu^{-1/2}h(U(\xi))^{1/2}\Phi(t,\xi)$. Then, using (2.3), we can rewrite the problem (1.1), (1.2) in the form

$$\Phi_t + Z(U)\Phi - \mu\Phi_{\xi\xi} = \mu^{1/2}h(U)^{-1/2}F(U,\phi_\xi), \tag{3.34}$$

$$\Phi(0,\xi) = \Phi_0(\xi), \tag{3.35}$$

where $Z(U)$ and Φ_0 are defined by (2.7) and $(3.3)_2$, respectively, and ϕ_ξ on the right-hand side of (3.34) is given by

$$\phi_\xi = \mu^{-1/2}h(U)^{1/2}[(2\mu)^{-1}(f'(U) - s)\Phi + \Phi_\xi].$$

Theorem 3.10. *Suppose that the conditions in Theorem* 3.2 *hold, and* $\Phi_0 \in H^2$. *Then there exists a positive constant* δ_6 *such that if* $\|\Phi_0\|_2 \le \delta_6$, *then the problem* (3.34), (3.35) *has a unique global solution* $\Phi(t,\xi)$ *satisfying*

$$\Phi \in C^0(0,\infty;H^2) \cap C^1(0,\infty;L^2), \quad \Phi_\xi \in L^2(0,\infty;H^2)\,, \tag{3.36}$$

and the decay estimate

$$e^{\theta t}\|\Phi(t)\|_2^2 + \int_0^t e^{\theta\tau}\|\Phi_\xi(\tau)\|_2^2 d\tau \le C\|\Phi_0\|_2^2 \tag{3.37}$$

holds for $t \ge 0$, *where* θ *is as in Theorem* 3.2.

Once Theorem 3.10 is proved, we can easily have the unique global solution of (1.1), (1.2) and also obtain the decay estimate (3.5). In fact, using Sobolev's embedding theorem, (2.3) and (3.33), we have

$$\begin{aligned}\sup_{x\in\mathbb{R}} |u(t,x) - U(x - st + x_0)| &= \sup_{\xi\in\mathbb{R}} |\phi_\xi(t,\xi)| \\ &= \sup_{\xi\in\mathbb{R}} \left| \left(\frac{1}{2\mu}(f'(U) - s)\Phi + \Phi_\xi \right) \mu^{-1/2} h(U)^{1/2} \right| \\ &\le C \sup_{\xi\in\mathbb{R}} (|\Phi| + |\Phi_\xi|) \\ &\le C\|\Phi(t)\|_2 \le Ce^{-\theta t/2}\,.\end{aligned}$$

Therefore, it is sufficient to prove Theorem 3.10 for our purpose. To do that, we may combine a local existence result together with *a priori* estimates in the same way as before. Thus we may only show the following:

Proposition 3.11. (*a priori* estimate) *Let* T *be a positive constant and* $\Phi(t,\xi)$ *be a solution of the problem* (3.34), (3.35) *satisfying*

$$\Phi \in C^0(0,T;H^2) \cap C^1(0,T;L^2)\,, \tag{3.38}$$

$$\Phi_\xi \in L^2(0,T;H^2)\,. \tag{3.39}$$

Then there exist positive constants δ_7 *and* $C = C(\delta_7)$ *independent of* T *such that if* $\sup_{0\le t\le T} \|\Phi(t)\|_2 \le \delta_7$, *then the estimate*

$$e^{\theta t}\|\Phi(t)\|_2^2 + \int_0^t e^{\theta\tau}\|\Phi_\xi(\tau)\|_2^2 d\tau \le C\|\Phi_0\|_2^2 \tag{3.40}$$

holds for $t \in [0,T]$, *where* θ *is as in Theorem* 3.2.

Proof. Multiplying (3.34) by $e^{\theta t}\Phi$, we have

$$\left(\frac{1}{2}e^{\theta t}\Phi^2\right)_t + \left(Z(U) - \frac{\theta}{2}\right)e^{\theta t}\Phi^2 - (\mu e^{\theta t}\Phi\Phi_\xi)_\xi + \mu e^{\theta t}\Phi_\xi^2$$
$$= e^{\theta t}\Phi F(U, \phi_\xi)\mu^{1/2}h(U)^{-1/2}. \tag{3.41}$$

By Lemma 2.3 and the assumption of θ, we have

$$Z(U) - \frac{\theta}{2} \geq \frac{1}{4\mu}(f'(u_+) - s)^2 - \frac{\theta}{2} > 0. \tag{3.42}$$

Since

$$|e^{\theta t}\Phi F(U, \phi_\xi)\mu^{1/2}h(U)^{-1/2}| \leq Ce^{\theta t}h(U)^{1/2}|\Phi|\{(f'(U) - s)^2|\Phi|^2 + |\Phi_\xi|^2\}$$
$$\leq Ce^{\theta t}|\Phi|(|\Phi|^2 + |\Phi_\xi|^2), \tag{3.43}$$

after substituting (3.42) and (3.43) into (3.41), we can obtain

$$e^{\theta t}\|\Phi(t)\|^2 + \int_0^t e^{\theta\tau}\|\Phi(\tau)\|_1^2 d\tau \leq C\|\Phi_0\|^2 \tag{3.44}$$

provided that $\overline{N}(t) \equiv \sup_{0\leq\tau\leq t} \|\Phi(\tau)\|_2$ is suitably small. Next, applying ∂_ξ to (3.34) and multiplying it by $e^{\theta t}\Phi_\xi$, we obtain after integration by parts

$$e^{\theta t}\|\Phi_\xi(t)\|^2 + \int_0^t e^{\theta\tau}\|\Phi_\xi(\tau)\|_1^2 d\tau$$
$$\leq C\left\{\|\Phi_{0,\xi}\|^2 + \int_0^t e^{\theta\tau}\|\Phi(\tau)\|_1^2 + \int_0^t \int e^{\theta\tau}|\Phi_{\xi\xi}||F(U, \phi_\xi)/h(U)^{1/2}|d\xi d\tau\right\}. \tag{3.45}$$

We estimate the last term on the right-hand side of (3.45) as

$$\int_0^t \int e^{\theta\tau}|\Phi_{\xi\xi}||F(U, \phi_\xi)/h(U)^{1/2}|d\xi d\tau$$
$$\leq C\int_0^t \int e^{\theta\tau}|\Phi_{\xi\xi}|(|\Phi|^2 + |\Phi_\xi|^2)d\xi d\tau$$
$$\leq C\overline{N}(t)\int_0^t e^{\theta\tau}\|\Phi(\tau)\|_2^2 d\tau. \tag{3.46}$$

Therefore substituting (3.46) into (3.45) and combining it with (3.44), we have

$$e^{\theta t}\|\Phi(t)\|_1^2 + \int_0^t e^{\theta\tau}\|\Phi(\tau)\|_2^2 d\tau \leq C\|\Phi_0\|_1^2, \tag{3.47}$$

for suitably small $\overline{N}(t)$. Similarly, applying ∂_ξ^2 to (3.34) and multiplying it by $e^{\theta t}\Phi_{\xi\xi}$, we finally have

$$e^{\theta t}\|\Phi(t)\|_2^2 + \int_0^t e^{\theta\tau}\|\Phi(\tau)\|_3^2 d\tau \le C\|\Phi_0\|_2^2\,, \tag{3.48}$$

for suitably small $\overline{N}(t)$, say $\overline{N}(t) \le \delta_7$. Thus the proof of Proposition 3.11 is complete. □

4. Asymptotic Stability (Degenerate Case)

In this section, we show that the asymptotic stability of shock profile holds even under the degenerate shock condition $(2.2)_2$ for a class of initial data.

Theorem 4.1. *Suppose that* (1.3), (2.1), $(2.2)_2$, (2.4) *and* (2.8) *hold. Let* $U(x-st)$ *be a shock profile connecting* $u_\pm$. *Suppose that* $u_0(x) - U(x)$ *is integrable over* $\mathbb{R}$ *and that* $\Phi_0(x)$ *is well defined by* $(3.3)_2$ *and is in* H^2. *Then there exists a positive constant* δ_8 *such that if* $\|\Phi_0\|_2 \le \delta_8$, *then the problem* (1.1), (1.2) *has a unique global solution* $u(t,x)$ *satisfying*

$$u - U \in C^0(0,\infty;H^1) \cap L^2(0,\infty;H^2)\,, \tag{4.1}$$

where $U = U(x-st+x_0)$ *is the shifted shock profile. Moreover, the solution verifies the asymptotic behavior*

$$\sup_{x\in\mathbb{R}} |u(t,x) - U(x-st+x_0)| \to 0 \quad \text{as } t \to +\infty\,. \tag{4.2}$$

Remark. In this degenerate case, by using Proposition 2.2, we can see the weight function $h(U(x))^{-1/2}$ in $(3.3)_2$ has a polynomial decay order $O(|x|^{-1})$ as $x \to +\infty$ and an exponential decay order $O(e^{-C_-|x|})$ $[C_- = (f'(u_-)-s)/2\mu > 0]$ as $x \to -\infty$, respectively. Therefore, the condition $\|\Phi_0\|_2 \le \delta_1$ implies that the initial function $\phi_0(x)$ must have at least the same decay order as the weight function has.

To show Theorem 4.1, we employ the same change of unknown function in (3.33), and rewrite the problem (1.1), (1.2) in the form (3.34), (3.35). Then what we need is to show the following results on the problem (3.34), (3.35).

Theorem 4.2. *Suppose that the conditions in Theorem* 4.1 *hold. Then there exists a positive constant* δ_9 *such that if* $\|\Phi_0\|_2 \le \delta_9$, *then the problem* (3.34), (3.35) *has a unique global solution* $\Phi(t,\xi)$ *satisfying*

$$\Phi \in C^0(0,\infty;H^2) \cap C^1(0,\infty;L^2), \quad \Phi_\xi \in L^2(0,\infty;H^2)\,, \tag{4.3}$$

and the estimate

$$\|\Phi(t)\|_2^2 + \int_0^t \|\sqrt{Z(U)}\Phi(\tau)\|^2 + \|\Phi_\xi(\tau)\|_2^2 d\tau \le C\|\Phi_0\|_2^2 \tag{4.4}$$

holds for $t \ge 0$ *where* $Z(U)$ *is as in* (2.7).

Once Theorem 4.2 is proved, we can easily have the unique global solution of (1.1), (1.2) and also obtain the asymptotic behavior (4.2). In fact, using Sobolev's embedding theorem (2.3) and (3.33), we have

$$\begin{aligned}\sup_{x\in\mathbb{R}}|u(t,x)-U(x-st+x_0)| &= \sup_{\xi\in\mathbb{R}}|\phi_\xi(t,\xi)| \\ &\le C\sup_{\xi\in\mathbb{R}}(|\Phi(t,\xi)|+|\Phi_\xi(t,\xi)|) \\ &\le C\|\Phi(t)\|_2^{1/2}\|\Phi_\xi(t)\|^{1/2} \le C\|\Phi_0\|_2^{1/2}\|\Phi_\xi(t)\|^{1/2}\,. \end{aligned} \tag{4.5}$$

Since we can prove $\|\Phi_\xi(t)\| \to 0$ as $t\to\infty$ by using the estimate (4.4) and Eq. (3.34), we can show the asymptotic behavior (4.2) by (4.5). Thus, to prove Theorem 4.2 and eventually Theorem 4.1, we only need to show the following *a priori* estimate because all the other arguments can be done in the same way as in the previous sections.

Proposition 4.3. (*a priori* estimate) *Let T be a positive constant, and $\Phi(t,\xi)$ be a solution of the problem* (3.34), (3.35) *satisfying*

$$\Phi\in C^0(0,T;H^2)\cap C^1(0,T;L^2),\quad \Phi_\xi\in L^2(0,T;H^2)\,. \tag{4.6}$$

Then there exist positive constants δ_{10} and C independent of T such that if $\sup_{0\le t\le T}\|\Phi(t)\|_2\le\delta_{10}$, then the estimate

$$\|\Phi(t)\|_2^2+\int_0^t\|\sqrt{Z(U)}\Phi(\tau)\|^2+\|\Phi_\xi(\tau)\|_2^2d\tau\le C\|\Phi_0\|_2^2 \tag{4.7}$$

holds for $t\in[0,T]$.

Proof. Multiplying (3.34) by Φ and integrating it over $[0,t]\times\mathbb{R}$, we have

$$\begin{aligned}&\|\Phi(t)\|^2+\int_0^t\int Z(U)\Phi^2(\tau,\xi)d\xi d\tau+\int_0^t\|\Phi_\xi(\tau)\|^2d\tau \\ &\quad\le C(\|\Phi_0\|^2+\int_0^t\int|F(U,\phi_\xi)/h(U)^{1/2}||\Phi|d\xi d\tau)\,.\end{aligned} \tag{4.8}$$

Making use of Taylor's formula and Lemma 2.3, we have

$$\begin{aligned}&|F(U,\phi_\xi)/h(U)^{1/2}||\Phi| \\ &\quad\le C\{(f'(U)-s)^2h(U)^{1/2}|\Phi|^2+h(U)^{1/2}|\Phi_\xi|^2\}|\Phi| \\ &\quad\le C(Z(U)|\Phi|^2+|\Phi_\xi|^2)|\Phi|\,.\end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8) gives

$$\begin{aligned}&\|\Phi(t)\|^2+(1-C\overline{N}(t))\int_0^t\int Z(U)\Phi^2(\tau,\xi)d\xi d\tau \\ &\quad+(1-C\overline{N}(t))\int_0^t\|\Phi_\xi(\tau)\|^2d\tau\le C\|\Phi_0\|^2\,.\end{aligned} \tag{4.10}$$

Hence, if we assume $\overline{N}(t)$ suitably small, say $\overline{N}(t) \le \delta_{10}$, we obtain

$$\|\Phi(t)\|^2 + \int_0^t \|\sqrt{Z(U)}\Phi(\tau)\|^2 + \|\Phi_\xi(\tau)\|^2 d\tau \le C\|\Phi_0\|^2 . \tag{4.11}$$

Finally, applying ∂_ξ^l $(l = 1, 2)$ to (3.34) and multiplying it by $\partial_\xi^l \Phi$, we can prove

$$\|\partial_\xi^l \Phi(t)\|^2 + \int_0^t \|\partial_\xi^{l+1}\Phi(\tau)\|^2 d\tau \le C\|\Phi_0\|_l^2 , \quad (l = 1, 2) , \qquad (4.12)_l$$

on the same line as before. We omit the details. Combining (4.11) together with $(4.12)_l$, we obtain (4.7). Thus the proof of Proposition 4.3 is complete. □

5. Remark

As in Ref. 5, we can replace the conditions (1.3) and (2.8) by

$$f''(u) \lesseqgtr 0 \quad \text{for} \quad u \gtrless 0 \quad \text{under consideration} , \tag{5.1}$$

$$f'''(u) < 0 \quad \text{for} \quad u \ne 0 \quad \text{under consideration} , \tag{5.2}$$

respectively. Then all the results in the previous sections are also valid. In fact, if u is a solution of (1.1) under the conditions (5.1) and (5.2), the change of independent variable, $y = -x$, transforms (1.1) into

$$u_t + \widetilde{f}(u)_y = \mu u_{yy} , \tag{5.3}$$

where $\widetilde{f}(u) = -f(u)$, and this $\widetilde{f}(u)$ verifies (1.3) and (2.8).

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