

Department of Mathematics and Statistics, Concordia University

Math 203: Calculus (I)

Midterm Exam, Winter Semester, 2007

Instructor: Dr. Ming Mei

NOTE: Calculators are not allowed.

1. [20pts]

- (a) Suppose $f(x) = \sqrt{1-x}$ and $g(x) = \sin^2 x$. Find $f \circ g \circ f$ and $g \circ f \circ g$. Simplify.
 (b) Find the inverse of the function $f(x) = 2 + x^5$. Determine the domain and the range of f and f^{-1} , respectively.

2. [20pts] Evaluate the limits.

$$(a) \quad \lim_{x \rightarrow 5} \frac{\sqrt{x-4} - 1}{x^2 - 4x - 5} \qquad (b) \quad \lim_{x \rightarrow \infty} \frac{4x(x^2 + 1)^4}{(3x^3 + 2)^2(2x^2 + 1)}$$

3. [20pts]

- (a) Consider the function $f(x) = \frac{|x-2|}{x-2}$. Calculate both one-sided limits at the point where the function is undefined.
 (b) Find parameters a and b such that the function

$$f(x) = \begin{cases} -\frac{4}{x^2}, & \text{if } x \leq -1, \\ ax - b, & \text{if } -1 < x \leq 0, \\ x^2 - 2, & \text{if } x > 0 \end{cases}$$

will be continuous at every point.

4. [40pts] Find derivatives of the following functions (do not simplify the answer):

- (a) $f(x) = \left(x + \frac{1}{x}\right)^2 \ln 5x$;
 (b) $f(x) = \frac{\sin^2 x}{1 + x^2}$;
 (c) $f(x) = \cos^2(x - e^{x^2})$;
 (d) $f(x) = x^{\arctan x^2}$ (use logarithmic differentiation).
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Solutions to the Midterm Test

(2)

by M. Mei

1(a).

$$\begin{aligned} f \circ g \circ f &= f(g(f(x))) \\ &= \sqrt{1 - g(f(x))} \\ &= \sqrt{1 - \sin^2(f(x))} \\ &= \sqrt{1 - \sin^2 \sqrt{1-x}} \\ &= \sqrt{\cos^2 \sqrt{1-x}} \\ &= |\cos \sqrt{1-x}|, \quad // \end{aligned}$$

$$\begin{aligned} f \circ f \circ g &= \sin^2(f(g(x))) \\ &= \sin^2 \sqrt{1 - g(x)} \\ &= \sin^2 \sqrt{1 - \sin^2 x} \\ &= \sin^2 \sqrt{\cos^2 x} \\ &= \sin^2 |\cos x|, \quad // \end{aligned}$$

1(b)

$$\begin{aligned} y = f(x) &= 2 + x^5 \\ \downarrow & \quad \quad \quad \downarrow \\ x &= 2 + y^5, \text{ i.e. } y = (x-2)^{\frac{1}{5}} \end{aligned}$$

So, the inverse: $f^{-1}(x) = \sqrt[5]{x-2}$

Domain of f : $(-\infty, \infty)$

Range of f : $(-\infty, \infty)$

Domain of f^{-1} : $(-\infty, \infty)$

Range of f^{-1} : $(-\infty, \infty)$

$$\begin{aligned}
 \underline{2(a)} \quad & \lim_{x \rightarrow 5} \frac{\sqrt{x-4} - 1}{x^2 - 4x - 5} = \lim_{x \rightarrow 5} \frac{\sqrt{x-4} - 1}{(x-5)(x+1)} \cdot \frac{\sqrt{x-4} + 1}{\sqrt{x-4} + 1} \\
 & = \lim_{x \rightarrow 5} \frac{(\sqrt{x-4})^2 - 1^2}{(x-5)(x+1)(\sqrt{x-4} + 1)} \\
 & = \lim_{x \rightarrow 5} \frac{\cancel{x-4} - 1}{(x-5)(x+1)(\sqrt{x-4} + 1)} \\
 & = \lim_{x \rightarrow 5} \frac{1}{(x+1)(\sqrt{x-4} + 1)} \\
 & = \frac{1}{(5+1)(\sqrt{5-4} + 1)} = \frac{1}{6 \cdot 2} = \frac{1}{12} //
 \end{aligned}$$

$$\begin{aligned}
 \underline{2(b)} \quad & \lim_{x \rightarrow \infty} \frac{4x(x^2+1)^4}{(3x^3+2)^2(2x^2+1)} \\
 & = \lim_{x \rightarrow \infty} \frac{4x(x^2+1)^4 / x^9}{(3x^3+2)^2(2x^2+1) / x^9} \quad \text{(Highest order is 9)} \\
 & = \lim_{x \rightarrow \infty} \frac{\frac{4x}{x} \cdot \frac{(x^2+1)^4}{x^8}}{\frac{(3x^3+2)^2}{x^6} \cdot \frac{(2x^2+1)}{x^2} \cdot \frac{1}{x}} \\
 & = \lim_{x \rightarrow \infty} \frac{4 \left(\frac{x^2+1}{x^2} \right)^4}{\left(\frac{3x^3+2}{x^3} \right)^2 \left(\frac{2x^2+1}{x^2} \right) \cdot \frac{1}{x}}
 \end{aligned}$$

$$= \lim_{x \rightarrow \infty} \frac{4 \left(1 + \frac{1}{x^2}\right)^4}{\left(3 + \frac{2}{x^3}\right)^2 \left(2 + \frac{1}{x^2}\right) \cdot \frac{1}{x}}$$

$$= \frac{4 (1+0)^4}{(3+0)^2 (2+0) \cdot 0} = \frac{4}{0} = \infty$$

"DNE"

3(a)

$$\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2} = \lim_{x \rightarrow 2^+} \frac{\cancel{x-2}}{\cancel{x-2}} = 1$$

$$\lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2} = \lim_{x \rightarrow 2^-} \frac{-(x-2)}{\cancel{x-2}} = -1$$

3(b)

Since $f(x)$ is needed to be continuous, we have:

$$f(-1^-) = f(-1^+), \text{ and } f(0^-) = f(0^+).$$

i.e.

$$\begin{cases} \frac{-4}{(-1)^2} = a(-1) - b \\ a \cdot 0 - b = 0^2 - 2 \end{cases}$$

which gives:

$$a = 2, \quad b = 2.$$

4(a)

$$f'(x) = \left(\left(x + \frac{1}{x} \right)^2 \right)' \ln 5x + \left(x + \frac{1}{x} \right)^2 (\ln 5x)'$$

(product rule)

$$= 2 \left(x + \frac{1}{x} \right)^{2-1} \cdot \left(x + \frac{1}{x} \right)' \ln 5x + \left(x + \frac{1}{x} \right)^2 \cdot \frac{1}{5x} \cdot (5x)'$$

(chain rule)

$$= 2 \left(x + \frac{1}{x} \right) (1 + (-1)x^{-2}) \ln 5x + \left(x + \frac{1}{x} \right)^2 \cdot \frac{1}{5x} \cdot 5$$

$$= 2 \left(x + \frac{1}{x} \right) \left(1 - \frac{1}{x^2} \right) \ln 5x + \left(x + \frac{1}{x} \right)^2 \cdot \frac{1}{x} \quad //$$

4(b)

$$f'(x) = \left(\frac{\sin^2 x}{1+x^2} \right)' = \frac{(\sin^2 x)'(1+x^2) - \sin^2 x(1+x^2)'}{(1+x^2)^2}$$

$$= \frac{2 \sin x (\sin x)' (1+x^2) - \sin^2 x (0+2x)}{(1+x^2)^2}$$

$$= \frac{2 \sin x \cos x (1+x^2) - 2x \sin^2 x}{(1+x^2)^2} \quad //$$

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$$\underline{4(c)}. \quad f'(x) = 2 [\cos(x - e^{x^2})]^{2-1} \cdot (\cos(x - e^{x^2}))'$$

(Chain Rule)

$$= 2 \cos(x - e^{x^2}) \cdot (-\sin(x - e^{x^2}))$$

$$\cdot (x - e^{x^2})'$$

(chain rule)

$$= -2 \cos(x - e^{x^2}) \sin(x - e^{x^2})$$

$$\cdot (1 - e^{x^2} \cdot (x^2)')$$

(chain rule)

$$= -2 \cos(x - e^{x^2}) \sin(x - e^{x^2})$$

$$\cdot (1 - 2x e^{x^2})$$

$$= -[\sin(2(x - e^{x^2}))] \cdot (1 - 2x e^{x^2})$$

//

4(d)

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$$y = x^{\arctan x^2}$$

Taking logarithms to the above equation:

$$\ln y = \ln x^{\arctan x^2}$$

$$= \arctan x^2 \cdot \ln x$$

Taking differentiation to the both sides of the above equation, we have:

$$\frac{1}{y} y' = (\arctan x^2)' \ln x + \arctan x^2 (\ln x)'$$

Chain rule

$$= \frac{1}{1+(x^2)^2} \cdot (x^2)' \ln x$$

$$+ \arctan x^2 \cdot \frac{1}{x}$$

$$= \frac{2x}{1+x^4} \ln x + \frac{1}{x} \arctan x^2$$

$$\text{So, } y' = y \left[\frac{2x}{1+x^4} \ln x + \frac{1}{x} \arctan x^2 \right]$$

$$= x^{\arctan x^2} \left[\frac{2x \ln x}{1+x^4} + \frac{\arctan x^2}{x} \right]$$

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