Review 3

CHAMPLAIN COLLEGE ST.-LAMBERT

MATH 201-NYB: Calculus II

Final Examination

Instructor: Dr. Ming Mei

Note: 1. Only scientific calculators are allowed;

1. (10pts) Sketch the graph of the function

$$f(x) = \begin{cases} -\sqrt{9-x^2}, & \text{if } -3 \le x < 0 \\ 3x-3, & \text{if } 0 < x \le 3, \end{cases}$$

then evaluate the definite integral $\int_{-3}^{3} f(x)dx$ by interpreting it in terms of area (do not antidifferentiate).

2. (28pts) Find the indefinite integrals:

(a) $\displaystyle\int x\, e^{x^2} dx$, (b) $\displaystyle\int (x+2)\ln x dx$, (c) $\displaystyle\int \frac{1}{x^2-2x-3}dx$, (d) $\displaystyle\int \frac{x}{\sqrt{4-x^2}}dx$.

3. (18pts)
   (a) Find the area bounded by the curves $y = x^2$ and $y = x$.
   (b) Find the volume of the solid which is obtained by rotating the region bounded by the curves $y = \sin x$ $(0 \le x \le \pi)$ and $y = 0$ about the x-axis.
   (c) Find the volume of the solid which is obtained by rotating the region bounded by the curves $y = \sin x$ $(0 \le x \le \pi)$ and $y = 0$ about the y-axis.

4. (7pts) Evaluate the given improper integral or show it to be divergent: $\int_2^\infty \frac{x^2}{x^3-1}dx$.

5. (6pts) Show the convergence or divergence of the following sequence: $\left\{\dfrac{\sin n}{n+1}\right\}$.

6. (18pts) Show the convergence or divergence of the following series:

$$\text{(a)} \sum_{n=2}^{\infty} \frac{n^2}{n^3-1}, \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n+1}, \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{2^{n+1}}{n^2 3^n}.$$

7. (7pts) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \dfrac{x^{n+1}}{(1+n)^2 3^n}$.

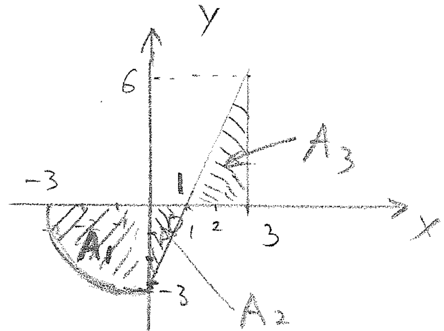
8. (6pts) Find Maclaurin series of $f(x) = \frac{x}{1+x}$.

Solutions

1. • For $-3 \le x < 0$, we have

$$y = f(x) = -\sqrt{9-x^2}$$

Taking square to the both sides of the above equation, we then have:



$$y^2 = \left(-\sqrt{9-x^2}\right)^2 = (-1)^2\left(\sqrt{9-x^2}\right)^2 = 9-x^2$$

i.e. $x^2+y^2=3^2$, a circle with center (0,0) and radius 3.

So, $y=-\sqrt{9-x^2}$ is the quarter of the circle in the III quadrant.

• For $0 < x \le 3$, $y = 3x-3$ is the straight line passing through $(0,-3)$, $(1,0)$ and $(3,6)$.

The graph is shown above, and

$$\int_{-3}^{3} f(x)\,dx = -A_1 - A_2 + A_3 = -\frac{1}{4}\pi\cdot 3^2 - \frac{1}{2}\cdot 1\cdot 3 + \frac{1}{2}\cdot 2\cdot 6$$

$$= \boxed{\frac{9}{2} - \frac{9}{4}\pi}$$

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2. (a) $\int x e^{-x^2}\,dx$

$= \int e^u \left(-\frac{1}{2}\,du\right)$

$= -\frac{1}{2}\int e^u\,du$

$= -\frac{1}{2} e^u + C$

$= \boxed{-\frac{1}{2}e^{-x^2} + C}$

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Substitution.

$u = -x^2$

$du = -2x\,dx$

i.e.

$x\,dx = -\frac{1}{2}du$

2(b). $\int (x+2) \ln x \, dx$

$= \int u\,dv = uv - \int v\,du$

$= (\ln x)(\frac{1}{2}x^2+2x) - \int (\frac{1}{2}x^2+2x)\frac{1}{x}dx$

$= (\frac{1}{2}x^2+2x)\ln x - \int (\frac{1}{2}x+2)\,dx$

$= \boxed{(\frac{1}{2}x^2+2x)\ln x - (\frac{1}{4}x^2+2x) + C}$ //

Integration by parts:

$u = \ln x$

$dv = (x+2)\,dx$

$du = \frac{1}{x}dx$

$v = \frac{1}{2}x^2+2x$

2(c) $\int \frac{1}{x^2-2x-3}dx = \int \frac{1}{(x+1)(x-3)}dx$

$= \int \left[\frac{\frac{1}{4}}{x-3} - \frac{\frac{1}{4}}{x+1}\right]dx$

$= \frac{1}{4}\ln|x-3| - \frac{1}{4}\ln|x+1| + C$

$= \boxed{\frac{1}{4}\ln\left|\frac{x-3}{x+1}\right| + C}$ //

Partial Fractions:

$\frac{1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

$= \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$

So

$A(x-3)+B(x+1) = 1$

Let $x=3$, then

$B(3+1) = 1 \Rightarrow \boxed{B = \frac{1}{4}}$

Let $x=-1$, then

$A(-1-3) = 1 \Rightarrow \boxed{A = -\frac{1}{4}}$

2(d) $\int \frac{x}{\sqrt{4-x^2}}dx$

$= \int \frac{1}{\sqrt{u}}(-\frac{1}{2}du)$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}du = -\frac{1}{2}\frac{u^{1-\frac{1}{2}}}{1-\frac{1}{2}} + C$

$= -\frac{1}{2}\frac{\sqrt{u}}{\frac{1}{2}} + C = \boxed{-\sqrt{4-x^2} + C}$ //

Substitution,

$u = 4-x^2$,

$du = -2x\,dx$

$x\,dx = -\frac{1}{2}du$

3(a)

$$A = \int_0^1 [\text{Top} - \text{Bottom}]\, dx$$
$$= \int_0^1 [x - x^2]\, dx$$
$$= \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_0^1$$
$$= \boxed{\tfrac{1}{6}}$$

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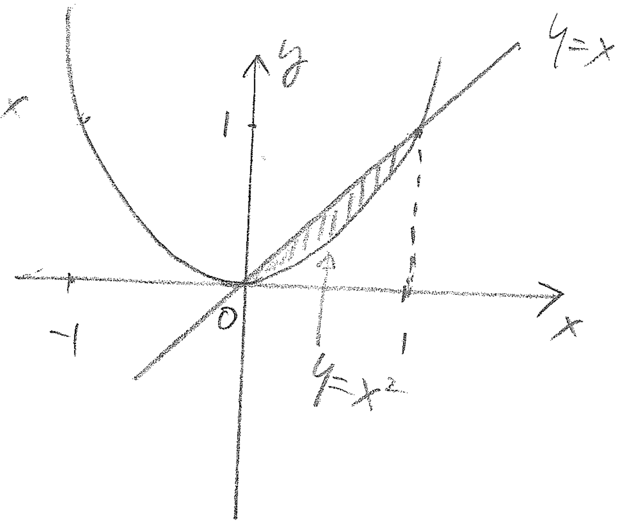

3(b)

$$V = \int_0^{\pi} \pi [f(x)]^2 dx$$
$$= \int_0^{\pi} \pi [\sin x]^2 dx$$
$$= \pi \int_0^{\pi} \sin^2 x\, dx$$
$$= \pi \int_0^{\pi} \frac{1 - \cos 2x}{2}\, dx$$
$$= \pi \int_0^{\pi} \frac{1}{2}\, dx$$
$$- \frac{\pi}{2} \int_0^{\pi} \cos 2x\, dx$$
$$= \frac{\pi}{2} x \Big|_0^{\pi} - \frac{\pi}{2} \int_0^{2\pi} \cos u \frac{du}{2}$$
$$= \frac{\pi}{2} \cdot (\pi - 0) - \frac{\pi}{2} \sin u \Big|_0^{2\pi}$$
$$= \frac{\pi^2}{2} - \frac{\pi}{2}[\sin 2\pi - \sin 0]$$
$$= \frac{\pi^2}{2} - \frac{\pi}{2}[0 - 0] = \boxed{\frac{\pi^2}{2}}$$

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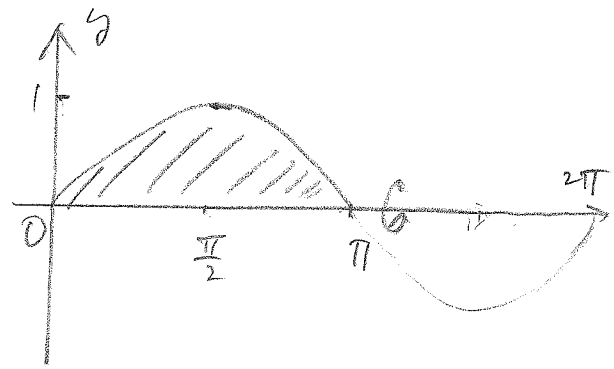
Substitute:
$u = 2x$
$du = 2dx$
$x = 0: \Rightarrow u = 0$
$x = \pi \Rightarrow u = 2\pi$



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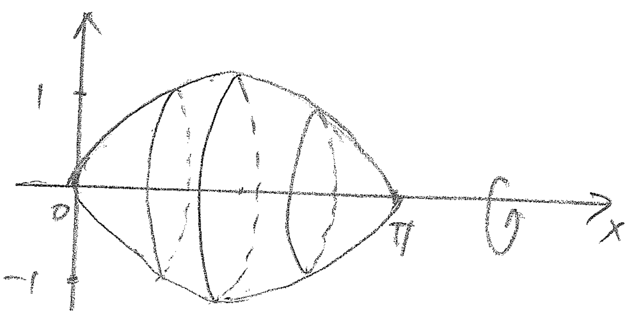
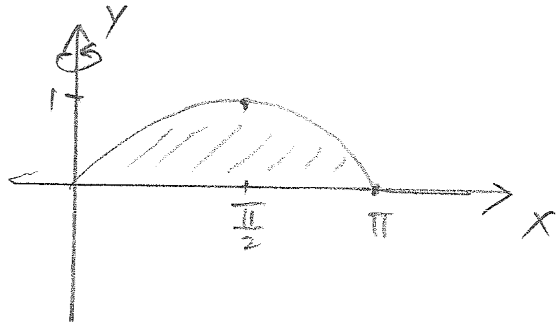


3(c)

$$V = \int_a^b 2\pi x f(x)\,dx$$



$$= 2\pi \int_0^{\pi} x \sin x \, dx$$

by Parts

$$= 2\pi \left[x(-\cos x)\Big|_0^{\pi} - \int_0^{\pi} (-\cos x)\,dx \right]$$

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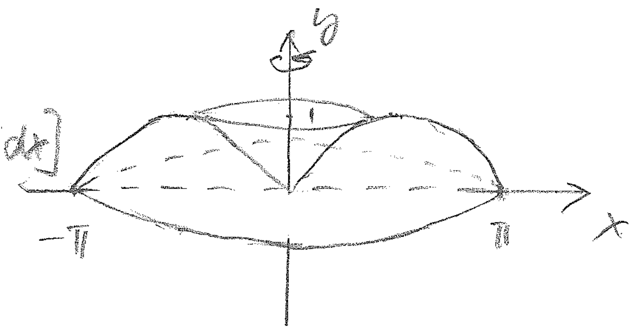

$$= 2\pi \left[-x\cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx \right]$$

$$= 2\pi \left[(-\pi \cos\pi + 0\cdot\cos 0) + \sin x \Big|_0^{\pi} \right]$$

$$= 2\pi \left[-\pi(-1) + 0 - (\sin\pi - \sin 0) \right]$$

$$= 2\pi \left[\pi + 0 - (0 - 0) \right]$$

$$= \boxed{2\pi^2}$$

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4. $$\int_2^{\infty} \frac{x^2}{x^3-1}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{x^2}{x^3-1}\,dx$$

Substitute:

$u = x^3 - 1$

$du = 3x^2 dx$

$x = 2, \Rightarrow u = 2^3 - 1 = 7$

$x = t \Rightarrow u = t^3 - 1$

$$= \lim_{t\to\infty} \int_7^{t^3-1} \frac{1}{u}\left(\frac{du}{3}\right)$$

$$= \frac{1}{3} \lim_{t\to\infty} \int_7^{t^3-1} \frac{1}{u}\,du$$

$$= \frac{1}{3} \lim_{t\to\infty} \ln|u| \Big|_7^{t^3-1} = \frac{1}{3} \lim_{t\to\infty} \left[\ln|t^3-1| - \ln 7 \right]$$

$= \infty$. So, it is divergent.

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5. Since $\sin n$ is bounded by

$$-1 \le \sin n \le 1$$

we have

$$-\frac{1}{n+1} \le \frac{\sin n}{n+1} \le \frac{1}{n+1}$$

Notice that

$$\lim_{n\to\infty}\left(-\frac{1}{n+1}\right)=0 \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n+1}=0,$$

by the Squeeze Theorem, we have

$$\lim_{n\to\infty}\frac{\sin n}{n+1}=0.$$

Namely, the sequence $\left\{\frac{\sin n}{n+1}\right\}$ is convergent to 0. //

6. (a). Let $a_n=\frac{n^2}{n^3-1}$, $b_n=\frac{n^2}{n^3}=\frac{1}{n}$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{n^2}{n^3-1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^2}{n^3-1}\times\frac{n}{1}$$

$$=\lim_{n\to\infty}\frac{n^3}{n^3-1}=\lim_{n\to\infty}\frac{n^3/n^3}{(n^3-1)/n^3}=\lim_{n\to\infty}\frac{1}{1-\frac{1}{n^3}}=\frac{1}{1-0}=1\ne 0.$$

By the Limit Comparison Test, both $\sum_{n=2}^{\infty}a_n=\sum_{n=2}^{\infty}\frac{n^2}{n^3-1}$ and $\sum_{n=2}^{\infty}b_n=\sum_{n=2}^{\infty}\frac{1}{n}$ have the same convergence or divergence. However, $\sum_{n=2}^{\infty}b_n=\sum_{n=2}^{\infty}\frac{1}{n}$ is divergent, because it is the p-series with $p=1$, therefore, $\sum_{n=2}^{\infty}\frac{n^2}{n^3-1}$ is also divergent.

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6(b). $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ is the alternating series

Let $a_n = \frac{1}{n+1}$, it can be easy to check:

- $a_n = \frac{1}{n+1} \geq \frac{1}{n+1+1} = a_{n+1}$

and

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$

By applying the Alternating Series Test, the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n+1}$ is convergent. //

6(c) Let $a_n = \frac{2^{n+1}}{n^2 3^n}$. Note that,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+2}}{(n+1)^2 3^{n+1}}}{\frac{2^{n+1}}{n^2 3^n}} = \lim_{n\to\infty} \frac{2^{n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{2^{n+1}}$$

$$= \lim_{n\to\infty} \frac{2^{\cancel{n}+2}}{2^{\cancel{n}+1}} \cdot \frac{\cancel{3^n}}{3^{\cancel{n}+1}} \cdot \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{2^2}{2^1} \cdot \frac{1}{3^1} \cdot \left(\frac{n}{n+1}\right)^2$$

$$= \lim_{n\to\infty} \frac{2}{3} \left(\frac{n/n}{(n+1)/n}\right)^2 = \lim_{n\to\infty} \frac{2}{3}\left(\frac{1}{1+\frac{1}{n}}\right)^2 = \frac{2}{3}\left(\frac{1}{1+0}\right)^2$$

$$= \frac{2}{3} < 1,$$

by the Ratio Test, it is convergent. //

7. Let $a_n = \dfrac{x^{n+1}}{(1+n)^2 3^n}$. then

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x^{n+2}|}{(2+n)^2 3^{n+1}} \Big/ \frac{|x|^{n+1}}{(1+n)^2 3^n}$$

$$= \lim_{n\to\infty} \frac{|x|^{n+2}}{|x|^{n+1}} \cdot \frac{(1+n)^2}{(2+n)^2} \cdot \frac{\cancel{3^n}}{3^{n+1}}$$

$$= \lim_{n\to\infty} |x| \left(\frac{1+n}{2+n}\right)^2 = |x| \lim_{n\to\infty} \left(\frac{(1+n)/n}{(2+n)/n}\right)^2$$

$$= |x| \lim_{n\to\infty} \left(\frac{\frac{1}{n}+1}{\frac{2}{n}+1}\right)^2 = |x| \left(\frac{0+1}{0+1}\right)^2 = |x|$$

By the Ratio test, when $L<1$, the series is convergent, Namely, $|x|<1$, i.e. $-1<x<1$, the series is convergent.

For the endpoints, when $x=-1$, we have

$\displaystyle\sum_{n=1}^{\infty} \frac{x^{n+1}}{(1+n)^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(1+n)^2 3^n}$ is alternating, and

$a_n = \dfrac{1}{(1+n)^2 3^n} \geq \dfrac{1}{(2+n)^2 3^{n+1}} = a_{n+1}$ is $\downarrow$

$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(1+n)^2 3^n} = \frac{1}{\infty} = 0.$

By the alternating series test, $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n+1}}{(1+n)^2 3^n}$ is convergent at $x=-1$.

when $x=1$, we have

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{(1+n)^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{(1+n)^2 3^n} \leq \sum_{n=1}^{\infty} \frac{1}{(1+n)^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, because it is the p-series with $p=2>1$, by the Comparison Test

$$\sum_{n=1}^{\infty} \frac{1}{(1+n)^2 3^n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

the series $\sum_{n=1}^{\infty} \frac{1}{(1+n)^2 3^n}$ is convergent.

Therefore, the interval of convergence is: $[-1,1]$

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8. Notice that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x|<1$$

then, the Maclaurin series of $f(x)=\frac{x}{1+x}$ is:

$$f(x) = \frac{x}{1+x} = x \cdot \frac{1}{1-(-x)} = x \cdot \sum_{n=0}^{\infty} (-x)^n$$

$$= x \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+1} \quad \text{for } |x|<1.$$

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