Final Review #1

Champlain College—St. Lambert

Review Questions for Final Exam

Math 201-NYB: Calculus II

Instructor: Dr. Ming Mei

MARKS

[10] **1.** (a) Sketch the graph of the function

$$f(x) = \begin{cases} -\sqrt{4-x^2}, & \text{if } -2 \le x < 0 \\ 2x-2, & \text{if } \ 0 \le x \le 3 \end{cases}$$

Evaluate the definite integral $\int_{-2}^{3} f(x)\,dx$ by interpreting it in terms of area (do not antidifferentiate).

(b) Find the derivative of the function

$$F(x) = \int_{\frac{\pi}{2}}^{\frac{x^2}{2}} \frac{\cos(2t^2)}{t^3+2t}\,dt$$

[15] **2.** Find the indefinite integrals:

(a) $\displaystyle\int \sin x\, e^{\cos x}\,dx$ (b) $\displaystyle\int x^2 \ln x\,dx$ (c) $\displaystyle\int \frac{1}{x^2-x-2}\,dx$

[15] **3.** Calculate the definite integrals:

(a) $\displaystyle\int_1^e \frac{2x+\ln^2 x}{x}\,dx$ (b) $\displaystyle\int_0^{\pi} (x^2+2)\cos 2x\,dx$ (c) $\displaystyle\int_0^2 x\sqrt{4-x^2}\,dx$

Review 1

[15] 4. (a) Find the area bounded by the curves $y = e^x$, $y = e$ and $x = 0$.

(b) Find the volume of the solid obtained by rotating the region bounded by the curves $y = \cos x$ $(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2})$ about the x-axis.

(c) Find the average value of the function $f(x) = \tan x$ on the interval $[0, \frac{\pi}{4}]$.

[12] 5. Evaluate the given improper integral or show that it diverges:

(a) $\displaystyle\int_2^\infty \frac{x^3}{x^4 - 1}\, dx$ (b) $\displaystyle\int_{\frac{1}{2}}^2 \frac{dx}{\sqrt{2x-1}}$

[9] 6. Find the limit of the sequence or show that it does not exist:

(a) $\left\{\dfrac{3n - 2n^3}{3n^3 - 2n}\right\}$ (b) $\{(-1)^n \cos \pi n\}$ (c) $\left\{\dfrac{(-1)^n n^2}{1 - 2n^2}\right\}$

[12] 7. Test each of the following series to determine if it is convergent or divergent:

(a) $\displaystyle\sum_{n=1}^\infty \frac{3^{n-1}}{n!}$ (b) $\displaystyle\sum_{n=2}^\infty \frac{n}{n^4 - 2}$ (c) $\displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{\sqrt[3]{n+1}}$

[12] 8. (a) Find the sum of the series $\displaystyle\sum_{n=3}^\infty \left(\frac{3}{2}\right)^{-n}$

(b) Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^\infty \frac{(x-3)^n}{n^2\, 2^n}$

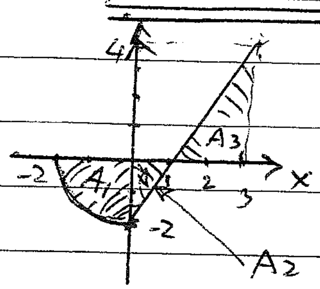
(c) Find the MacLaurin series for the function $f(x) = xe^{\frac{x^2}{2}}$

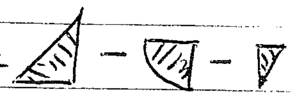
MATH NYB : Cal (II)

Instructor: M. Mei

Solutions to Review 1

1. (a)



$\int_{-2}^{3} f(x)\,dx =$ 

$= \frac{1}{2}\cdot 2\cdot 4 - \frac{1}{4}\cdot\pi\cdot 2^2 - \frac{1}{2}\cdot 1\cdot 2 = \boxed{3-\pi}$

(b) $F'(x) = \dfrac{\cos\left(2\left(\frac{x^2}{2}\right)^2\right)}{\left(\frac{x^2}{2}\right)^3 + 2\cdot\frac{x^2}{2}}\cdot\left(\frac{x^2}{2}\right)' = \boxed{\dfrac{8\cos\frac{x^4}{2}}{x^5+8x}}$

2. (a) $\int \sin x\, e^{\cos x}\,dx \xrightarrow[du=-\sin x\,dx]{u=\cos x} -\int e^u\,du = -e^u + c$

$= \boxed{-e^{\cos x} + c}$

(b) $\int x^2 \ln x\,dx$ $\qquad$ $u = \ln x \quad dv = x^2 dx$; $\quad du = \frac{1}{x}dx \quad v = \frac{x^3}{3}$

$= (\ln x)\frac{x^3}{3} - \int \frac{x^3}{3}\cdot\frac{1}{x}dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx$

$= \boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + c}$

(c) $\int \frac{1}{x^2-x-2}\,dx = \int \frac{1}{(x-2)(x+1)}\,dx = \int \left[\frac{\frac{1}{3}}{x-2} - \frac{\frac{1}{3}}{x+1}\right]dx$

$= \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + c = \boxed{\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + c}$

3. (a) $\int_1^e \frac{2x+\ln^2 x}{x}\,dx = \int_1^e \left[2 + \frac{\ln^2 x}{x}\right]dx = \int_1^e 2\,dx + \int_1^e \frac{\ln^2 x}{x}\,dx$

$= 2x\Big|_1^e + \int_0^1 u^2\,du$ $\qquad$ $u = \ln x, \quad du = \frac{1}{x}dx$

$= 2x\Big|_1^e + \frac{u^3}{3}\Big|_0^1$ $\qquad$ $x=1 \Rightarrow u = \ln 1 = 0$; $\quad x = e \Rightarrow u = \ln e = 1$

$= \boxed{2(e-1) + \frac{1}{3}}$

(b) $\int_0^{\pi} (x^2+2)\cos 2x\,dx$

$= (x^2+2)\frac{1}{2}\sin 2x\Big|_0^{\pi} - \int_0^{\pi} 2x\cdot\frac{1}{2}\sin 2x\,dx$

$= -\int_0^{\pi} x\sin 2x\,dx$

$= -x\left(-\frac{1}{2}\cos 2x\right)\Big|_0^{\pi} + \int_0^{\pi}\left(-\frac{1}{2}\cos 2x\right)dx$

$= \frac{\pi}{2} - \frac{1}{2}\int_0^{\pi}\cos 2x\,dx = \boxed{\frac{\pi}{2}}$

$u = x^2+2 \qquad dv = \cos 2x\,dx$

$du = 2x\,dx \qquad v = \frac{1}{2}\sin 2x$

Again, integration by parts

$u = x \qquad dv = \sin 2x\,dx$

$du = dx \qquad v = -\frac{1}{2}\cos 2x$

(c) $\int_0^2 x\sqrt{4-x^2}\,dx$

$= \int_4^0 \sqrt{u}\left(-\frac{1}{2}du\right) = -\frac{1}{2}\int_4^0 u^{\frac{1}{2}}du$

$= \frac{1}{2}\int_0^4 u^{\frac{1}{2}}du = \frac{1}{2}\,\frac{u^{3/2}}{3/2}\Big|_0^4$

$= \boxed{\frac{8}{3}}$

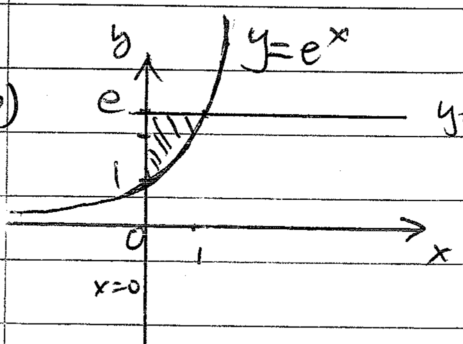
$u = 4-x^2,\ du = -2x\,dx$

$x\,dx = -\frac{1}{2}du$

$x=0 \Rightarrow u=4$

$x=2 \Rightarrow u=0$

4 (a)



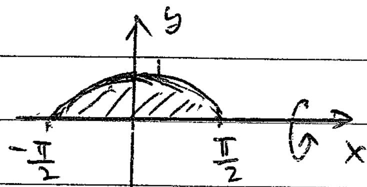
$A = \int_0^1 [\text{Top} - \text{Bottom}]\,dx$

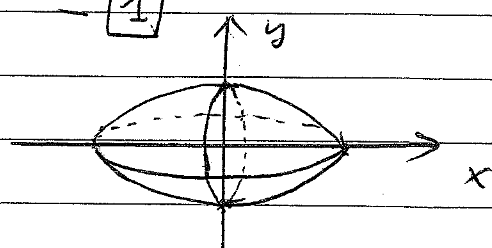
$= \int_0^1 [e - e^x]\,dx$

$= (ex - e^x)\Big|_0^1 = (e\cdot 1 - e^1) - (0 - e^0)$

$= \boxed{1}$

(b)





$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi[\cos x]^2\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\cos^2 x\,dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\cdot\frac{1+\cos 2x}{2}\,dx = \left(\frac{\pi}{2}x + \frac{\sin 2x}{4}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{\frac{\pi^2}{2}}$

(c) $f_{average} = \frac{1}{\frac{\pi}{4}-0}\int_0^{\frac{\pi}{4}} \tan x\, dx = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\, dx$

$= \frac{4}{\pi}\int_1^{\frac{1}{\sqrt{2}}} \frac{-1}{u}\, du = \frac{4}{\pi}\int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{u} du$ | $u = \cos x$, $du = -\sin x\, dx$; $x=0 \Rightarrow u = \cos 0 = 1$; $x = \frac{\pi}{4} \Rightarrow u = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$= \frac{4}{\pi} \ln u \Big|_{\frac{1}{\sqrt{2}}}^{1} = \frac{4}{\pi} \ln\sqrt{2}$

$= \boxed{\frac{2}{\pi}\ln 2}$

5 (a) $\int_2^\infty \frac{x^3}{x^4-1} dx = \lim_{t\to\infty}\int_2^t \frac{x^3}{x^4-1} dx$ | $u = x^4-1$; $du = 4x^3 dx$; $x=2 \Rightarrow u = 7$; $x = t \Rightarrow u = t^3-1$

$= \lim_{t\to\infty}\int_7^{t^3-1} \frac{1}{u}\cdot\frac{du}{4} = \frac{1}{4}\lim_{t\to\infty}\int_7^{t^3-1} \frac{1}{u} du$

$= \lim_{t\to\infty} \left[\ln(t^3-1) - \ln 7\right] = \boxed{\infty}$

So, it is divergent.

(b) $\int_{\frac{1}{2}}^{2} \frac{dx}{\sqrt{2x-1}} = \lim_{t\to\frac{1}{2}^+}\int_t^2 \frac{dx}{\sqrt{2x-1}}$ | $u = 2x-1$; $du = 2dx$; $x = t \Rightarrow u = 2t-1$; $x = 2 \Rightarrow u = 3$

$= \lim_{t\to\frac{1}{2}^+}\int_{2t-1}^{3} \frac{1}{\sqrt{u}}\frac{1}{2}du$

$= \lim_{t\to(1/2)^+}\left[\sqrt{3} - \sqrt{2t-1}\right] = \boxed{\sqrt{3}}$

6 (a) $\lim_{n\to\infty} \frac{3n-2n^3}{3n^3-2n} = -\frac{2}{3}$. (b) $\lim_{n\to\infty} (-1)^n \cos n\pi = 1$

(c) doesn't exit. because, when n is even, $\lim_{n\to\infty}\frac{(-1)^n n^2}{1-2n^2} = -\frac{1}{2}$

when n is odd $\lim_{n\to\infty}\frac{(-1)^n n^2}{1-2n^2} = \frac{1}{2}$.

7(a) By the Ratio Test,

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{3^n}{(n+1)!}\Big/\frac{3^{n-1}}{n!}=\lim_{n\to\infty}\frac{3}{n+1}=0<1$$

So, the series is Convergent.

(b) $\because \frac{n}{n^4-2}\sim\frac{1}{n^3}$, and $\lim_{n\to\infty}\frac{n}{n^4-2}\Big/\frac{1}{n^3}=1$

So, $\sum_{n=1}^{\infty}\frac{n}{n^4-2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^3}$ have the same convergence or divergence. Note that, $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent, because it is a p-series with $p=3>1$. So, $\sum_{n=1}^{\infty}\frac{n}{n^4-2}$ is also Convergent.

(c) it is an alternating series, and $a_n=\frac{1}{\sqrt[3]{n+1}}\downarrow$

$\lim_{n\to\infty}\frac{1}{\sqrt[3]{n+1}}=0$. By the Alternating series, it is convergent.

8(a) $$\sum_{n=3}^{\infty}\left(\frac{3}{2}\right)^{-n}=\sum_{n=3}^{\infty}\left(\frac{2}{3}\right)^n=\cancel{\sum_{n=0}}\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^4+\left(\frac{2}{3}\right)^5+\cdots$$

$$=\left(\frac{2}{3}\right)^3\left[1+\left(\frac{2}{3}\right)^1+\left(\frac{2}{3}\right)^2+\cdots\right]$$

$$=\left(\frac{2}{3}\right)^3\cdot\frac{1}{1-\frac{2}{3}}=\frac{8}{27}\cdot\frac{1}{\frac{1}{3}}=\boxed{\frac{8}{9}}$$

(b) By the Ratio Test, we need

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+1)^2 2^{n+1}}\Big/\frac{(x-3)^n}{n^2 2^n}\right|$$

$$= \lim_{n\to\infty} \frac{|x-3|\; n^2}{2\;(n+1)^2} = \frac{|x-3|}{2} \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \frac{|x-3|}{2} < 1,$$

i.e. $|x-3| < 2 \iff 1 < x < 5$

for the convergance of the power series.

Furthermore, when $x = 5$:

$$\sum_{n=1}^{\infty} \frac{(5-3)^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is}$$

Convergent, because of p-series with $p=2$.

When $x = 1$:

$$\sum_{n=1}^{\infty} \frac{(1-3)^n}{n^2 2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} \text{ is an alternating}$$

series, and $\frac{1}{n^2} \searrow$, $\lim_{n\to\infty} \frac{1}{n^2} = 0$. So

by the alternating Test, it is convergent.

Therefore, the convergent interval is: $\boxed{[1,5]}$

(c) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$

So, $f(x) = x e^{\frac{x^2}{2}} = x \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!}$

//