CHAMPLAIN COLLEGE ST.-LAMBERT

MATH 201-NYB: Calculus II

Review Questions for Test # 3

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1. Test the convergence or divergence of the sequence. If it is convergent, find its limit.

   A). $a_n = \dfrac{2+n^3}{1+2n^3}$, B). $a_n = \dfrac{9^{n+1}}{10^n}$,

   C). $a_n = \dfrac{n \sin n}{n^2+1}$, D). $a_n = \dfrac{n}{\ln n}$.

2. Determine whether the series is convergent or divergent.

   A). $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$, B). $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$,

   C). $\displaystyle\sum_{n=0}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$, D). $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 2^n}$.

3. Find the interval of convergence for the power series.

   A). $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{4n^{\frac{n}{2}}}$, B). $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{4^{\frac{n}{2}} n}$.

4. Find Maclaurin series for the function $f(x) = \frac{\ln(1+x)}{x}$.

Solutions

Q1.

A).

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2+n^3}{1+2n^3} = \lim_{n\to\infty} \frac{(2+n^3)/n^3}{(1+2n^3)/n^3} = \lim_{n\to\infty} \frac{\frac{2}{n^3}+1}{\frac{1}{n^3}+2} = \frac{1}{2}.$$

So, it is convergent, and the limit is $\frac{1}{2}$.

B).

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{9^{n+1}}{10^n} = \lim_{n\to\infty} \frac{9\cdot 9^n}{10^n} = 9 \lim_{n\to\infty} \left(\frac{9}{10}\right)^n = 9\cdot 0 = 0.$$

So, it is convergent, and the limit is 0.

C). Since $\sin n$ is bounded by $-1 \le \sin n \le 1$, we have

$$-\frac{n}{n^2+1} \le \frac{n\sin n}{n^2+1} \le \frac{n}{n^2+1}.$$

Taking the limit as $n \to \infty$ to the above inequalities, and noting that $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$, then by using the Squeeze theorem, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n\sin n}{n^2+1} = 0.$$

So, it is convergent, and the limit is 0.

D). By using the L'Hospital's Rule, we otain

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{(n)'}{(\ln n)'} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty.$$

So, it is divergent.

Q2.

A). Note that $\frac{n}{n^3+1} \le \frac{n}{n^3} = \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, because it is a p-series with $p = 2 > 1$, then by applying the Comparison Test, the series $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ is also convergent.

Another method is the Limit Comparison Test. Note that $\frac{n}{n^3+1} \sim \frac{n}{n^3} = \frac{1}{n^2}$. Let $a_n = \frac{n}{n^3+1}$, $b_n = \frac{1}{n^2}$. Since

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^3+1}\Big/\frac{1}{n^2} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1,$$

by the Limit Comparison Test, both the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ have the same convergence or divergence. Note that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is the p-series with $p = 2$, and is convergent, therefore, $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ is also convergent.

B). $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is an alternating series, and the general term $a_n = \frac{1}{\sqrt{n+1}}$, obviously, is decreasing to 0, then by using the Alternating Test, it is convergent.

C). Let $a_n = \frac{n^{2n}}{(1+n^2)^n}$. Applying the Root Test,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n^{2n}}{(1+2n^2)^n}} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2}{1+2n^2}\right)^n} = \lim_{n\to\infty} \frac{n^2}{1+2n^2} = \frac{1}{2} < 1,$$

so it is convergent.

D). Let $a_n = \frac{(-5)^{2n}}{n^2 2^n}$. Applying the Ratio Test,

$$\begin{aligned} L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-5)^{2(n+1)}}{(n+1)^2 2^{n+1}} \Big/ \frac{(-5)^{2n}}{n^2 2^n}\right| = \lim_{n\to\infty} \left|\frac{(-5)^{2(n+1)}}{(n+1)^2 2^{n+1}} \cdot \frac{n^2 2^n}{(-5)^{2n}}\right| \\ &= \lim_{n\to\infty} \frac{(-5)^2}{2} \cdot \frac{n^2}{(n+1)^2} = \frac{25}{2} > 1, \end{aligned}$$

so it is divergent.

Q3.

A). Obviously, the series is equivalent to $\sum_{n=1}^{\infty} \frac{(x-1)^n}{4n^{\frac{n}{2}}} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^{\frac{n}{2}}}$. Let $a_n = \frac{(x-1)^n}{n^{n/2}} = \left(\frac{x-1}{\sqrt{n}}\right)^n$. Using the Root Test,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{x-1}{\sqrt{n}}\right)^n\right|} = \lim_{n\to\infty} \frac{|x-1|}{\sqrt{n}} = |x-1| \lim_{n\to\infty} \frac{1}{\sqrt{n}} = |x-1| \cdot 0 = 0 < 1, \quad \text{for all } x,$$

the series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^{\frac{n}{2}}}$ is convergent for all x, so is $\sum_{n=1}^{\infty} \frac{(x-1)^n}{4n^{\frac{n}{2}}}$. Thus, the convergence interval is $I = (-\infty, \infty)$.

B). Let $a_n = \frac{(x-1)^n}{4^{n/2} n} = \frac{(x-1)^n}{2^n n}$. Applying the Ratio Test,

$$\begin{aligned} L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{2^{n+1}(n+1)} \Big/ \frac{(x-1)^n}{2^n n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{(x-1)^n}\right| \\ &= \lim_{n\to\infty} \frac{|x-1|n}{2(n+1)} = \frac{|x-1|}{2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x-1|}{2}, \end{aligned}$$

if $L = \frac{|x-1|}{2} < 1$, i.e., $-1 < x < 3$, the series is convergent.

On the hand, for $x = -1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1-1)^n}{4^{n/2} n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which is alternating. Note that $\frac{1}{n}$ is decreasing and $\lim_{n\to\infty}\frac{1}{n}=0$, then the Alternating Test implies that the series is convergent for $x=-1$.

For $x=3$, the series becomes

$$\sum_{n=1}^{\infty}\frac{(3-1)^n}{4^{n/2}n}=\sum_{n=1}^{\infty}\frac{(2)^n}{2^n n}=\sum_{n=1}^{\infty}\frac{1}{n},$$

which is divergent, because it is the p-series with $p=1$.

Therefore, the convergent interval is $I=[-1,3)$.

Q4. Since

$$\frac{1}{1-x}=1+x+x^2+\cdots=\sum_{n=0}^{\infty}x^n,$$

then

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n,$$

and

$$\ln(1+x)=\int\frac{1}{1+x}dx=\int\sum_{n=0}^{\infty}(-1)^n x^n dx=\sum_{n=0}^{\infty}(-1)^n\int x^n dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}.$$

Therefore,

$$\frac{\ln(1+x)}{x}=\frac{\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}}{x}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^n.$$