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Global solutions for bistable degenerate reaction–diffusion equation with time-delay and nonlocal effect

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ABSTRACT

In this paper, we prove the existence, uniqueness and regularity of global solution to the bistable reaction–diffusion equation with degenerate diffusion and nonlocal time-delay. The adopted approach is the Holmgren's approximation scheme combining the compactness analysis.

1. Introduction

This paper concerns the nonlocal degenerate diffusion equation with time-delay, a model of population dynamics for single species [1–3]

$$\begin{cases} u_t(x,t) - \Delta u^m(x,t) = g(u(x,t), J * S(u(x,t-r))), & t \geq 0,\ x \in \mathbb{R}, \\ u(x,s) = u_0(x,s), & s \in [-r,0],\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with the convolution

$$J * S(u(x,t-r)) = \int_{\mathbb{R}} J(y) S(u(x-y,t-r)) dy. \tag{1.2}$$

Here, $u(t,x)$ denotes the population density of single species at time t and location x, $\Delta u^m(x,t)$ with $m > 1$ is the degenerate diffusion, and $g(\cdot,\cdot)$ is the nonlinear function involving the death rate function and the birth rate function, where $r > 0$ is the matured age of the single species, the so-called time-delay. Throughout the

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paper, we assume that the kernel $J(\cdot)$ is positive and unit, and Eq. (1.1) satisfies the bistable conditions, namely,

(H1) $J(y) \geq 0$ for $y \in \mathbb{R}$, and $\int_{\mathbb{R}} J(y)dy = 1$;
(H2) $g(u,v) \in C^2([0,1] \times [S(0), S(1)], \mathbb{R})$, $g_2(u,v) \geq 0$ for $(u,v) \in [0,1] \times [S(0), S(1)]$, $S(u) \in C^2([0,1], \mathbb{R})$ and $S'(u) \geq 0$ for $u \in [0,1]$;
(H3) $g(0, S(0)) = g(1, S(1)) = 0$, $g_1(1, S(1)) + g_2(1, S(1))S'(1) < 0$ and $g_1(0, S(0)) + g_2(0, S(0))S'(0) < 0$.

Here, $g_1(u,v) = \frac{\partial}{\partial u} g(u,v)$, $g_2(u,v) = \frac{\partial}{\partial v} g(u,v)$, and $S'(w) = \frac{d}{dw} S(w)$.

The typical models of $g(u,v)$ satisfying $(H_2) - (H_3)$ are as follows:

- Neural network [4–6] with $g(u,v) = -\alpha u + v$ for $\alpha > 1$:

$$u_t(x,t) - \Delta u^m(x,t) = -\alpha u + \int_{-\infty}^{\infty} J(x-y) S(u(y, t-r)) dy, (t,x) \in \mathbb{R}^+ \times \mathbb{R}, r > 0.$$

- Hodgkin–Huxley [7,8] with $g(u,v) = u(1-u)v$, $S(u) = u - a$ for $a \in (0,1)$:

$$u_t(x,t) - \Delta u^m(x,t) = u(1-u) \int_{-\infty}^{\infty} J(x-y)(u(y,t-r) - a) dy, (t,x) \in \mathbb{R}^+ \times \mathbb{R}, r > 0.$$

Eq. (1.1) possesses some challenging features caused by the degenerate diffusion, time-delay and the nonlocal effect, because the degenerate diffusion usually causes the solutions to lose their regularity, the large time-delay usually makes the solutions to be oscillating, and the nonlocal effect enhances the complexity of calculation. So it is worth to investigate the well-posedness of the solutions and the structure of the solutions, particularly, the traveling wave solutions. In this short note, we study the existence, uniqueness and regularity of the global solutions to (1.1), and leave the structure of sharp traveling waves for future.

For the nonlinear porous media equations without time-delay, the properties of the solutions have been extensively studied in [3,7–13]. The new phenomenon for sharp traveling waves with critical wave speed was first observed by Aronson [14] for Fisher–KPP equations, then further studied by De Pablo and Vazuquez [15] and Malaguti and Marcelli [16] for the generalized Fisher–KPP equations with degeneracy of diffusion. The exact form of sharp traveling waves was obtained by Gilding and Kersner [17] in the case of $m = 2$. When these Fisher–KPP (mono-stable type) equations are affected by the time-delay, the sharp waves with or without oscillations were classified by Xu et al. [3,18,19] based on the sizes of time-delay and the wave speed, and the global stability of the non-sharp traveling waves were studied by Huang et al. [11] and Liu-Mei-Yang [12], recently. The asymptotic behavior of solutions for the Dirichlet boundary case were studied in [20–22].

For the regular Huxley (bistable type) reaction–diffusion equations, the smooth and monotone traveling waves were intensively investigated in [4–6,23–26]. When these bistable reaction–diffusion equations with degeneracy of diffusion and time-delay, the existence of sharp traveling waves was proved by Jin et al. [7,8].

However, for the case of the bistable reaction–diffusion equations with the effect of degenerate diffusion, time-delay and nonlocality, the relevant study is almost nothing as we know. As the first step, we are going to prove the existence, uniqueness and regularity of global solution for (1.1). This is the main issue of the present note. The adopted approach is the Holmgren's iteration scheme with the help of the compactness estimates and the monotonic technique. The crucial steps of the Holmgren's iteration scheme are to denote a monotonic function for constructing an auxiliary equation and establish the estimates for the solutions of the adjoint equation. Then by upper and lower solutions to iterate.

2. Main result and proof

Definition 2.1. A function $u \in L^2_{loc}(\mathbb{R}_+ \times \mathbb{R})$ is called a weak solution of (1.1) with (1.2) if $0 \le u \le 1$, $\nabla u^m \in L^2_{loc}(\mathbb{R}_+ \times \mathbb{R})$, and for any $T > 0$ and $\psi \in C_0^\infty((-r,T) \times \mathbb{R})$,

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}} u(x,t)\psi_t dxdt + \int_0^T \int_{\mathbb{R}} \nabla u^m(x,t) \cdot \nabla \psi dxdt \\ &= \int_{\mathbb{R}} u_0(x,0)\psi(0,x)dx + \int_0^T \int_{\mathbb{R}} g(u(x,t), J * S(u(x,t-r)))\psi dxdt. \end{aligned}$$

Definition 2.2. A function u is called an upper (lower) solution of (1.1) if the following inequality holds in the sense of distributions,

$$u_t(t,x) - (u^m)_{xx}(t,x) - g(u(t,x), J * S(u(t-r,x))) \ge (\le) 0. \tag{2.1}$$

Theorem 2.1. *Assume that (H_1)–(H_3) hold. Let the initial data $u_0(s,x) \in L^\infty([-r,0] \times \mathbb{R})$ satisfy*

$$0 < u_0(s,x) < 1, (s,x) \in [-r,0] \times \mathbb{R}, \tag{2.2}$$

and

$$\lim_{x \to -\infty} u_0(s,x) = 0, \quad \lim_{x \to +\infty} u_0(s,x) = 1, \textit{uniformly in } s \in [-r,0]. \tag{2.3}$$

Then there exists a uniquely global solution of (1.1) with Hölder continuity $C^{\frac{1}{2m},\frac{1}{4m}}(\mathbb{R}_+ \times \mathbb{R})$ such that $u \in \mathcal{D}$, and

$$0 < u(t,x) < 1, (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

and

$$\lim_{x \to -\infty} u(t,x) = 0, \quad \lim_{x \to +\infty} u(t,x) = 1, \ uniformly \ in \ t \in \mathbb{R}_+, \tag{2.4}$$

where

$$\begin{aligned} \mathcal{D} = \{u | u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap C^{\frac{1}{2m},\frac{1}{4m}}(\mathbb{R}_+ \times \mathbb{R}); (u^m)_x \in L^\infty(\mathbb{R}_+; L^2_{loc}(\mathbb{R})); \\ (u^m)_t \in L^2_{loc}(\mathbb{R}_+ \times \mathbb{R})\}. \end{aligned}$$

For the local equation

$$\begin{cases} u_t(x,t) - \Delta u^m(x,t) = g(u(x,t), S(u(x,t-r))), & t \ge 0, \ x \in \mathbb{R}, \\ u(x,s) = u_0(x,s), & s \in [-r,0], \ x \in \mathbb{R}, \end{cases} \tag{2.5}$$

we have the similar result.

Theorem 2.2. *Assume that $(H_2) - (H_3)$ hold. Let the initial data $u_0(s,x) \in L^\infty([-r,0] \times \mathbb{R})$ satisfies (2.2)* and *(2.3). Then there exists a uniquely global solution of (2.5) with Hölder continuity $C^{\frac{1}{2m},\frac{1}{4m}}(\mathbb{R}_+ \times \mathbb{R})$ such that $u \in \mathcal{D}$ and*

$$0 < u(t,x) < 1, (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

and

$$\lim_{x \to -\infty} u(t,x) = 0, \quad \lim_{x \to +\infty} u(t,x) = 1, \ uniformly \ in \ t \in \mathbb{R}_+.$$

Since the proof of Theorem 2.2 is pretty similar to Theorem 2.1, therefore we just need to prove Theorem 2.1. Note that Eq. (1.1) is degenerate at $u = 0$, so we will adopt the Holmgren's approximation scheme and compactness method in this section to treat it.

Let us define a sufficiently smooth sequence $u_0^l(s,x)$ with $l \gg 1$ such that $0 < u_0^l(s,x) < 1$ for $(s,x) \in [-r,0] \times [-l,l]$ and $\lim_{l\to\infty} u_0^l(s,x) = u_0(s,x)$ uniformly in x and s. Now we consider the initial–boundary-value problem as follows:

$$\begin{cases} u_t(x,t) - \Delta u^m(x,t) = g(u(x,t), J * S(u(x,t-r))), & t \geq 0,\ x \in [-l,l], \\ u(-l,t) = \frac{1}{l}, u(l,t) = 1 - \frac{1}{l}, & t \geq 0, \\ u(x,s) = u_0^l(s,x), (s,x) \in [-r,0] \times [-l.l]. & \end{cases} \tag{2.6}$$

We state the definitions of the upper and lower solutions of (2.6) and some lemmas before we prove the existence of the approximate solutions to (2.6).

Definition 2.3. A function $u \in C(\mathbb{R}_+ \times [-l,l])$ with $u^m \in W^{1,2}_{loc}(\mathbb{R}_+ \times [-l,l])$ is called a upper (lower) solution of (2.6), if it satisfies

$$\begin{cases} u_t(x,t) - \Delta u^m(x,t) \geq (\leq) g(u(x,t), J * S(u(x,t-r))), & t \geq 0,\ x \in [-l,l], \\ u(-l,t) \geq (\leq) \frac{1}{l}, u(l,t) \geq (\leq) 1 - \frac{1}{l}, & t \geq 0, \\ u(x,s) \geq (\leq) u_0^l(s,x), & (s,x) \in [-r,0] \times [-l,l]. \end{cases}$$

Let $T > 0$ be a given positive constant and $Q_T := [0,T] \times [-l,l]$, define a space

$$\begin{aligned} C_+([0,T]; H^1([-l,l])) = \{ v(t,x) | v(t,x) > 0 \ for \ (t,x) \in Q_T \ and \\ v \in C([0,T]; H^1([-l,l])) \}. \end{aligned} \tag{2.7}$$

For $v(t,x) \in C_+([0,T]; H^1([-l,l]))$, denote

$$F(v) := K_1 v(t,x) + g(v, J * S(v_r)),\ v_r := v(t-r,x), \tag{2.8}$$

where

$$K_1 = \sup_{(u,v)\in[0,1]\times[S(0),S(1)]} |g_1(u,v)|. \tag{2.9}$$

For $0 < v_1(t,x) \leq v_2(t,x)$, $(t,x) \in [-r,T] \times [-l,l]$,

$$\begin{aligned} & F(v_2) - F(v_1) \\ & = K_1(v_2 - v_1) + g(v_2, J * S(v_{2r})) - g(v_1, J * S(v_{2r})) + g(v_1, J * S(v_{2r})) \\ & \quad - g(v_1, J * S(v_{1r})) \\ & = K_1(v_2 - v_1) + g_1(\theta_1, J * S(v_{2r}))(v_2 - v_1) + g_2(v_1, J * S(\theta_2)) S'(\theta_3) J * (v_{2r} - v_{1r}) \\ & = (K_1 + g_1(\theta_1, J * S(v_{2r})))(v_2 - v_1) + g_2(v_1, J * S(\theta_2)) S'(\theta_3) J * (v_{2r} - v_{1r}) \\ & \geq 0,\ (t,x) \in Q_T,\ \theta_i \in [0,1],\ i = 1,2,3, \end{aligned}$$

which follows from the definition of K_1 and (H_2). Let $v_1 = 0$, then $F(v) \geq 0$ for $(t,x) \in Q_T$. We first pay attention to the following problem

$$\begin{cases} u_t(x,t) - \Delta u^m(x,t) + K_1 u(t,x) = F(v), & t \geq 0,\ x \in [-l,l], \\ u(-l,t) = \frac{1}{l}, u(l,t) = 1 - \frac{1}{l}, & t \geq 0, \\ u(x,s) = u_0^l(s,x), & (s,x) \in [-r,0] \times [-l,l]. \end{cases} \tag{2.10}$$

From [12], the following lemmas hold.

Lemma 2.3 (*[12]*). *Let $u(t,x)$ be the solution of (2.10) with $0 < v(t,x) < 1$ for $(t,x) \in Q_T$ and $0 < u_0^l(s,x) < 1$ for $(s,x) \in [-r,0] \times [-l,l]$. Then*

$$0 < u(t,x) < 1, \;\; (t,x) \in Q_T. \tag{2.11}$$

Lemma 2.4 (*[12]*). *Let $0 < v_1(t,x) \le v_2(t,x) < 1$ for $(t,x) \in [-r,T] \times [-l,l]$ and $u_1(t,x)$ and $u_2(t,x)$ be the solution of (2.10) with respect to $v_1(t,x)$ and $v_2(t,x)$, respectively. Then*

$$0 < u_1(t,x) \le u_2(t,x) < 1, \;\; (t,x) \in Q_T. \tag{2.12}$$

Lemma 2.5. *For any given $f(t,x) \in C_0^\infty(Q_T)$, where $Q_T := [0,T] \times [-l,l]$ with $l > 1$, $\epsilon \in (0,1)$, $\eta \in (0,1)$, and $0 \le A(t,x) \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, $B(t,x) \in C^1(\mathbb{R})$ is bounded. Let $\psi_{\epsilon,\eta}(t,x)$ be the solution of the following parabolic problem*

$$\begin{cases} \psi_t + (A_\epsilon(t,x) + \eta)\psi_{xx} + B_\epsilon \psi = f(t,x), & (t,x) \in Q_T, \\ \psi(t,-A) = 0, \;\; \psi(t,A) = 0, & t \in [0,T], \\ \psi(T,x) = 0, & x \in [-l,l], \end{cases} \tag{2.13}$$

where $A_\epsilon(t,x)$ and $B_\epsilon(t,x)$ are the smooth approximations of $A(t,x)$ and $B(t,x)$ respectively, such that

$$A(t,x) \le A_\epsilon(t,x) \le A(t,x) + \epsilon, \;\; \lim_{\epsilon \to 0} B_\epsilon(t,x) = B(t,x). \tag{2.14}$$

Then $\psi_{\epsilon,\eta}(t,x)$ satisfies

$$\sup_{(t,x) \in Q_T} |\psi_{\epsilon,\eta}(t,x)| \le C, \;\; \iint_{Q_T} (A_\epsilon(t,x) + \eta) \left(\frac{\partial^2 \psi_{\epsilon,\eta}(t,x)}{\partial x^2} \right)^2 dxdt \le C\eta^{-1}, \tag{2.15}$$

and

$$\iint_{Q_T} \left(\frac{\partial \psi_{\epsilon,\eta}(t,x)}{\partial x} \right)^2 dxdt \le C\eta^{-1}, \;\; \sup_{t \in [0,T]} \int_{-l}^{l} \left| \frac{\partial \psi_{\epsilon,\eta}(t,x)}{\partial x} \right| dx \le C, \tag{2.16}$$

where C denotes a generic constant which is independent of η and ϵ, and may take different values in different cases.

Proof. The proof can be similarly done as shown in Lemma 3.2.1 and Lemma 3.2.2 of [13], so we omit its detail.

Now, applying Lemmas 2.3–2.5 to Eq. (2.10), we can prove the existence of the solution for (2.6). □

Lemma 2.6. *Let $u_0^l(s,t) \in C_+([-r,0]; H^1([-l,l]))$ satisfy $0 < u_0^l < 1$. Then the problem (2.6) admits a unique smooth solution $u(t,x) \in C_+([0,T]; H^1([-l,l]))$ for any $T > 0$, satisfying*

$$0 < u(t,x) < 1, \;\; (t,x) \in [0,T] \times [-l,l]. \tag{2.17}$$

Proof. From Definition 2.3, $\overline{u} := 1$ and $\underline{u} := 0$ are a pair of upper and lower solutions of (2.6). Note that, for given function v, the IBVP (2.10) defines an operator $\mathcal{P}$ such that the solution of (2.10) can be written in the form of $u = \mathcal{P}(v)$. Thus, by initially setting $v = \underline{u}^0$ and $v = \overline{u}^0$, respectively, we can construct two sequences $\{\underline{u}^n\}$ and $\{\overline{u}^n\}$ by $\underline{u}^n = \mathcal{P}(\underline{u}^{n-1})$ and $\overline{u}^n = \mathcal{P}(\overline{u}^{n-1})$, for $n = 1, 2, \ldots$, respectively.

Thanks to Lemmas 2.3 and 2.4, we can prove

$$0 = \underline{u}^0 < \underline{u}^1 \le \cdots \le \underline{u}^n \le \cdots \le \overline{u}^n \le \cdots \le \overline{u}^1 < \overline{u}^0 = 1.$$

Thus there exist two limit functions $\hat{u}(t,x)$ and $\check{u}(t,x)$ with respect to $\{\underline{u}^n\}$ and $\{\overline{u}^n\}$ as $n \to \infty$, respectively, satisfying

$$0 < \hat{u}(t,x) \le \check{u}(t,x) < 1, \ \ (t,x) \in [0,T] \times [-l,l]. \tag{2.18}$$

Next we need to prove $\hat{u}(t,x) = \check{u}(t,x)$ as the unique solution of (2.6). Let $w = \check{u}(t,x) - \hat{u}(t,x)$, then

$$\begin{cases} w_t - \partial_x^2(A(t,x)w) = g(\check{u}, J * S(\check{u}_r)) - g(\hat{u}, J * S(\hat{u}_r)), & (t,x) \in [0,T] \times [-l,l], \\ w(t,-l) = 0, w(t,l) = 0, & t > 0, \\ w(s,x) = 0, & (s,x) \in [-r,0] \times [-l,l], \end{cases} \tag{2.19}$$

where

$$A(t,x) = m \int_0^1 (\theta\check{u} + (1-\theta)\hat{u})^{m-1} d\theta.$$

We note that $0 < A(t,x) \in C([0,T] \times [-l,l])$, due to (2.18).

For $t \in [0,r]$, $t - r \in [-r,0]$,

$$g(\check{u}, J * S(\check{u}_r)) - g(\hat{u}, J * S(\hat{u}_r)) = g(\check{u}, J * S(u_0^l)) - g(\hat{u}, J * S(u_0^l)) =: B(t,x)w,$$

where $B(t,x) := \int_0^1 g_1(\theta\check{u} + (1-\theta)\hat{u}, J * S(u_0^l)) d\theta$. Since $|g_1(u,v)| < K_1$ for $(u,v) \in [0,1] \times [S(0), S(1)]$, $|B(t,x)| < K_1$, then w satisfies

$$\begin{cases} w_t - \partial_x^2(A(t,x)w) - B(t,x)w = 0, & (t,x) \in [0,T] \times [-l,l], \\ w(t,-l) = 0, \ \ w(t,l) = 0, & t > 0, \\ w(s,x) = 0, & (s,x) \in [-r,0] \times [-l,l]. \end{cases} \tag{2.20}$$

By the definition of weak solutions, for any test function $\psi \in C^\infty([0,r] \times [-l,l])$, the solution of (2.20) satisfies

$$\iint_{Q_r} w(\psi_t + A(t,x)\psi_{xx} + B(t,x)\psi) dxdt = 0, \tag{2.21}$$

where $Q_r := [0,r] \times [-l,l]$.

Since $A(t,x)$ and $B(t,x)$ are merely bounded and measurable, it is difficult to analyze the solvability of the adjoint problem directly. Based on this reason, for any given $f(t,x) \in C_0^\infty(Q_r)$, we focus on the following approximation adjoint problem

$$\begin{cases} \psi_t + (A_\epsilon(t,x) + \eta)\psi_{xx} + B_{\epsilon,\eta}^\delta \psi = f(t,x), & (t,x) \in Q_r, \\ \psi(t,-l) = 0, \ \ \psi(t,l) = 0, & t \in [0,r], \\ \psi(r,x) = 0, & x \in [-l,l]. \end{cases} \tag{2.22}$$

Here, $\delta > 0$ and $\eta > 0$ are sufficiently small constants, for $|\check{u} - \hat{u}| \ge \delta$, $A(t,x) \ge L(\delta) > 0$, let

$$b_\eta^\delta = \begin{cases} (A(t,x) + \eta)^{-\frac{1}{2}} B, & |\check{u} - \hat{u}| \ge \delta, \\ 0, & |\check{u} - \hat{u}| < \delta, \end{cases}$$

thus $A_\epsilon(t,x)$ and $b_{\epsilon,\eta}^\delta$ are the smooth approximations of $A(t,x)$ and b_η^δ, and $B_{\epsilon,\eta}^\delta = b_{\epsilon,\eta}^\delta (A_\epsilon + \eta)^{\frac{1}{2}}$.

We take the solution $\psi_{\epsilon,\eta}$ of (2.22) as the test function in (2.21), then

$$\begin{aligned} \iint_{Q_r} wf dxdt &= \iint_{Q_r} w(\psi_t + (A_\epsilon + \eta)\psi_{xx} + B_{\epsilon,\eta}^\delta \psi) dxdt \\ = \iint_{Q_r} w\eta\psi_{xx} dxdt &+ \iint_{Q_r} w(A_\epsilon - A)\psi_{xx} dxdt + \iint_{Q_r} w(B_{\epsilon,\eta}^\delta - B)\psi dxdt. \end{aligned}$$

From [13], let $\epsilon \to 0$, $\eta \to 0$, $\delta \to 0$, then the right-hand side of the above equation tends to zero. By the arbitrariness of f, then $w = 0$ for $(t,x) \in Q_r$, namely, $\check{u} = \hat{u}$ for $t \in [0,r]$.

Repeating the above procedure to $[r, 2r]$, $[2r, 3r]$, $\ldots$, we obtain $\check{u} = \hat{u}$ for $t \in [0,T] \times [-l,l]$. The proof is complete. □

Proof of Theorem 2.1. Let $a \in (-l+2, l-2)$, and $\eta(x) \in C_0^\infty((a-2, a+2))$ be a function satisfying

$$0 \le \eta(x) \le 1;\ |\eta'(x)| \le 1;\ \eta(x) = 1,\ x \in (a-1, a+1).$$

Similar to [12], by Young's inequality, the mean value theorem of integrals and $0 < u < 1$, it is easy to deduce

$$\int_t^{t+\sigma} \int u^{m-1} |u_s|^2 \eta^4 dxds + \sup_t \int |(u^m)_x|^2 \eta^4 dx \le C, \tag{2.23}$$

where C is independent of l. We denote the weak limit of u^l by u as $l \to \infty$. The solutions of (1.1) satisfy $0 \le u \le 1$ and $u^m \in L^\infty(\mathbb{R}_+; H^1_{loc}(\mathbb{R}))$. The solutions also satisfy $(u^m)_t \in L^2((t, t+\delta) \times (a-1, a+1))$. For any $t_2 \le t_1 \in \mathbb{R}_+, x \in \mathbb{R}$, take a ball B_r of radius r centered at x, where $r = |t_1 - t_2|^{\frac{1}{2}}$. By the Poincaré inequality and the mean value theorem, there exists $x^* \in B_r$ such that

$$|u^m(t_1, x^*) - u^m(t_2, x^*)| \le C|t_1 - t_2|^{\frac{1}{2}} r^{-\frac{1}{2}}.$$

Then

$$|u^m(t_1, x) - u^m(t_2, x)| \le C|t_1 - t_2|^{\frac{1}{4}},$$

which implies that $u^m \in C^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R})$. Furthermore, $u \in C^{\frac{1}{4m}, \frac{1}{2m}}(\mathbb{R}_+ \times \mathbb{R})$.

For the uniqueness of original solution, let u_1 and u_2 be two solutions of (1.1), and denote $v = u_1 - u_2$. When $t \in [0, r]$, $u_1(t-r, x) = u_2(t-r, x) = u_0(t-r, x)$, we note that

$$\begin{cases} v_t - \partial_x^2(A(t,x)v) - B(t,x)v = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ \lim_{x \to \pm\infty} v(t,x) = 0, & t \in [0, r], \\ v(s,x) = 0, & (s,x) \in [-r, 0] \times \mathbb{R}, \end{cases}$$

where

$$A(t,x) := m \int_0^1 (\theta u_1 + (1-\theta) u_2)^{m-1} d\theta,$$

and

$$B(t,x) := \int_0^1 g_1(\theta u_1 + (1-\theta) u_2, J * S(u_0)) d\theta.$$

From the definition of weak solution, the weak form for the above equation is reduced to

$$\int_0^r \int_{\mathbb{R}} u(t,x) \Big[\frac{\partial \psi}{\partial t} + DA(t,x) \frac{\partial^2 \psi}{\partial x^2} + B(t,x)\psi \Big] dxdt = 0, \tag{2.24}$$

with any test function $\psi \in C_0^\infty([-r, T] \times \mathbb{R})$. For any smooth functions $g(x,t) \in C_0^\infty(R \times (0, r))$, let B_{R_0} be the ball with radius $R_0 > 0$ such that supp $g \subset B_{R_0}$. For any $R > R_0 + 1$, $\eta > 0$, and $\varepsilon > 0$, $A_\varepsilon(t,x)$ and $B_\varepsilon(t,x)$ are the smooth approximations of $A(t,x)$ and $B(t,x)$ respectively, we consider the following adjoint problem in $C_0^\infty([0, r] \times \mathbb{R})$:

$$\begin{cases} \dfrac{\partial \psi}{\partial t} + (DA_\varepsilon(t,x) + \eta) \dfrac{\partial^2 \psi}{\partial x^2} + B_\varepsilon(t,x)\psi = -g(x,t), & |x| < R,\ t \in (0, r], \\ \frac{\partial \psi}{\partial \nu} = 0, & |x| = R,\ t \in (0, r], \\ \psi(x, r) = 0, & |x| < R. \end{cases} \tag{2.25}$$

Let $\psi(t,x) = \psi^R_{\varepsilon,\eta}(t,x)$ be the solution of (2.25) with zero extension to the whole space of $\mathbb{R}$, and let $\xi_R(x) \in C_0^\infty(\mathbb{R})$ be a cut-off function such that $0 \le \xi_R(\xi) \le 1$, $\xi_R(x) = 1$ for $|x| \le R-1$, and $\xi_R(x) = 0$ for $|x| > R - \frac{1}{2}$. Particularly, by taking $\phi(t,x) = \xi_R(x)\psi^R_{\varepsilon,\eta}(t,x)$ as the test function in (2.24), using Lemma 2.5, by the same fashion of Proposition 3.2 in [12], we derive

$$\int_0^r \int_{\mathbb{R}} u(t,x) g(x,t) dxdt = 0.$$

Repeating the same procedure and using the arbitrariness of $g(x,t)$, we can prove the uniqueness. The proof of Theorem 2.1 is complete. □

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References

[1] J.D. Murry, Mathematical Biology I: An Introduction, Springer-Verlag, New York, 2002.
[2] J.W.-H. So, J. Wu, X. Zou, A reaction–diffusion model for a single species with age structure. I. travelling wavefronts on unbounded domains, Proc. R. Soc. Lond. Ser. A 457 (2001) 1841–1853.
[3] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Travelling waves for time-delayed reaction diffusion equations with degenerate diffusion, J. Differential Equations 265 (2018) 4442–4485.
[4] P.C. Bressloff, Spatiotemporal dynamics of continuum neural fields, J. Phys. A 45 (2012) 033001, 109 pp.
[5] B. Ermentrout, J.B. McLeod, Existence and uniqueness of travelling waves for a neural network, Proc. Roy. Soc. Edinburgh Sect. A 123 (1993) 461–478.
[6] S. Ma, J. Wu, Existence, uniqueness and asymptotic stability of traveling wavefronts in a non-local delayed diffusion equation, J. Dynam. Differential Equations 19 (2007) 391–436.
[7] C.H. Jin, J.X. Yin, Traveling waves for a time delayed non-Newtonian filtration equation, Physica D 241 (2012) 1789–1803.
[8] C.H. Jin, J.X. Yin, S.N. Zheng, Traveling waves for a time delayed newtonian filtration equation, J. Differential Equations 254 (2013) 1–29.
[9] P. Bénilan, M.G. Crandall, M. Pierre, Solutions of the porous medium equation in $\mathbb{R}^N$ under optimal conditions on initial values, Indiana Univ. Math. J. 33 (1984) 51–87.
[10] F. Sánchez-Garduño, P.K. Maini, Travelling wave phenomena in non-linear diffusion degenerate Nagumo equations, J. Math. Biol. 35 (1997) 713–728.
[11] R. Huang, C.H. Jin, M. Mei, J.X. Yin, Existence and stability of traveling waves for degenerate reaction–diffusion equation with time delay, J. Nonlinear Sci. 28 (2018) 1011–1042.
[12] C.C. Liu, M. Mei, J.Q. Yang, Global stability of traveling waves for nolocal time-delayed degenerate diffusion equation, J. Differential Equations In press.
[13] Z. Wu, J. Zhao, J. Yin, H. Li, Nonlinear Diffusion Equations, World Scientific Publishing Co. Put. Ltd., 2001.
[14] D.G. Aronson, Density-dependent interaction-diffusion systems, in: Dynamics and Modelling of Reactive Systems, in: Publ. Math. Res. Center Univ. Wisconsin, vol. 44, Academic Press, New York, London, 1980, pp. 161–176.
[15] A. de Pablo, J.L. Vazquez, Travelling waves and finite propagation in a reaction–diffusion equation, J. Differential Equations 93 (1991) 19–61.
[16] L. Malaguti, C. Marcelli, Sharp profiles in degenerate and doubly degenerate Fisher–KPP equations, J. Differential Equations 195 (2003) 471–496.
[17] B. Gilding, R. Kersner, A Fisher/KPP-type equation with density-dependent diffusion and convection: Travelling-wave solutions, J. Phys. A 38 (2005) 3367–3379.
[18] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Sharp oscillatory traveling waves of structured population dynamics model with degenerate diffusion, J. Differential Equations 269 (2020) 8882–8917.
[19] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Variational approach of critical sharp front speeds in degenerate diffusion model with time delay, Nonlinearity 33 (2020) 4013–4029.
[20] S. Ji, M. Mei, Z. Wang, Dirichlet problem for the Nicholson's blowflies equation with density-dependent diffusion, Appl. Math. Lett. 103 (2020) 106191, 7 pp.
[21] J.W.-H. So, Y.J. Yang, Dirichlet problem for the diffusive Nicholson's blowflies equation, J. Differential Equations 150 (1998) 317–348.
[22] J. Yang, M. Mei, Y. Wang, Novel convergence to steady-state for Nicholson's blowflies equation with Dirichlet boundary, Appl. Math. Lett. 114 (2021) 106895, 7 pp.
[23] X.F. Chen, Existence, uniqueness and asymptotic stability of traveling waves in nonlocal evolution equations, Adv. Differential Equations 2 (1997) 125–160.
[24] Y. Morita, H. Ninomiya, Entire solutions with merging fronts to reaction–diffusion equations, J. Dynam. Differential Equations 18 (2006) 841–861.
[25] Z.C. Wang, W.T. Li, S.G. Ruan, Entire solutions in bistable reaction–diffusion equations with nonlocal delayed nonlinearity, Trans. Amer. Math. Soc. 361 (2009) 2047–2084.
[26] H. Yagisita, Backward global solutions characterizing annihilation dynamics of travelling fronts, Publ. Res. Inst. Math. Sci. 39 (2003) 117–164.