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STRUCTURAL STABILITY OF RADIAL INTERIOR SUBSONIC STEADY-STATES TO N-D EULER-POISSON SYSTEM OF SEMICONDUCTOR MODELS WITH SONIC BOUNDARY*

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Abstract. This paper focuses on the structural stability of interior subsonic steady-states to the multi-dimensional Euler–Poisson system of hydrodynamic model for semiconductors with sonic boundary. As we know, the doping profile plays a crucial role for the existence/nonexistence of all types of subsonic/supersonic/transonic solutions for the multidimensional steady hydrodynamic model of semiconductors with sonic boundary. So it is quite significant and important to further investigate the structural stability of these physical solutions, when the doping profile is with small perturbation. The main issue of this paper is to study the structural stability of the radial interior subsonic steady-states, once the perturbations of doping profiles are small enough. Owing to the boundary degeneracy, and the strong singularity of the subsonic steady-states at the sonic boundary, it is full of challenges to derive the globally structural stability of interior subsonic solutions in multidimensional case. The adopted approach is the local singularity analysis at the sonic boundaries, with the help of the monotonicity argument.

Key words. structural stability, radial interior subsonic steady-states, hydrodynamic model for semiconductors, sonic boundary

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1. Introduction.

The hydrodynamic model for semiconductors, first introduced by Bløtekjær [8], is usually used to characterize the motion of the charged particles, such as the electrons and holes in semiconductors devices [19, 25]. The governing equations are following n-dimensional Euler-Poisson equations:

$$
\begin{cases}
\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\
(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \rho \nabla \Phi - \dfrac{\rho \mathbf{u}}{\tau}, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\
\Delta \Phi = \rho - b(x).
\end{cases}
\tag{1.1}
$$

Here the unknown functions $\rho(x,t)$, $\mathbf{u}(x,t)$, and $\Phi(x,t)$ represent the electron density, the velocity, and the electrostatic potential, respectively. The known function $p(\rho)$ is the pressure-density relation. For isentropic flows, $p(\rho) = \kappa \rho^\gamma$ with $\kappa > 0$ and $\gamma > 1$, and for isothermal flows, $p(\rho) = T\rho$ with the constant temperature $T > 0$. In the present paper, for simplicity but without loss of generality, we consider the

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isothermal case, and take $T = 1$ without loss of generality, namely, $p(\rho) = \rho$. The function $b(x) > 0$ is the doping profile which stands for the density of impurities in the semiconductor device. The constant $\tau > 0$ denotes the momentum relaxation time.

In this paper, we consider the following stationary equations of (1.1) in an annulus domain $\mathcal{A}$. Let the electric field $E := \nabla\Phi$, then we have corresponding stationary equations of (1.1) as follows:

$$\begin{cases} \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{\nabla\rho}{\rho} = E - \dfrac{\mathbf{u}}{\tau}, \quad x \in \mathcal{A}, \\ \operatorname{div}E = \rho - b(x), \end{cases} \tag{1.2}$$

where the annulus domain $\mathcal{A}$ is defined by

$$\mathcal{A} := \{x \in \mathbb{R}^n \mid r_0 < |x| < r_1\} \quad \text{with fixed constants} \quad 0 < r_0 < r_1,$$

the inner boundary and outer boundary of $\mathcal{A}$ are denoted by Γ_0 and Γ_1, respectively, namely,

$$\Gamma_0 := \{x \in \mathbb{R}^n \mid |x| = r_0\} \quad \text{and} \quad \Gamma_1 := \{x \in \mathbb{R}^n \mid |x| = r_1\},$$

and the closure of $\mathcal{A}$ is denoted by $\overline{\mathcal{A}} := \Gamma_0 \cup \mathcal{A} \cup \Gamma_1$.

From the terminology of fluid dynamics, we call $c(\rho) := \sqrt{p\prime(\rho)} = 1$ the local sound speed and $M := \frac{|\mathbf{u}|}{c(\rho)}$ the Mach number. Thus, the stationary flow of (1.2) is called to be subsonic if $M < 1$, supersonic if $M > 1$, and sonic if $M = 1$.

We assume $b(x) := \tilde{b}(r)$ in $\mathcal{A}$ with $\tilde{b}(r) \in L^\infty((r_0, r_1))$ and $r = |x|$. In the polar coordinates, we denote

$$(\rho, \mathbf{u}, E)(x) := (\tilde{\rho}(r), \tilde{u}(r)\vec{\mathbf{e}}, \tilde{E}(r)\vec{\mathbf{e}}), \tag{1.3}$$

where $\vec{\mathbf{e}} = \frac{x}{r}$ is a unit vector. The system (1.2) is subjected to the boundary value conditions:

$$(\rho|_{\Gamma_0}, \rho|_{\Gamma_1}, \rho\mathbf{u}|_{\Gamma_0}) = (\tilde{\rho}(r_0), \tilde{\rho}(r_1), \tilde{\rho}(r_0)\tilde{u}(r_0)\vec{\mathbf{e}}) = (\rho_0, \rho_1, j_0\vec{\mathbf{e}}) \tag{1.4}$$

for positive constants (ρ_0, ρ_1, j_0). Then, from (1.3), the system (1.2) with boundary value conditions (1.4) can be rewritten as the following boundary value problem:

$$\begin{cases} (r^{n-1}\tilde{\rho}\tilde{u})_r = 0, \\ (r^{n-1}\tilde{\rho}\tilde{u}^2)_r + r^{n-1}\tilde{\rho}_r = r^{n-1}\tilde{\rho}(\tilde{E} - \frac{\tilde{u}}{\tau}), \quad r \in (r_0, r_1), \\ (r^{n-1}\tilde{E})_r = r^{n-1}(\tilde{\rho} - \tilde{b}(r)), \\ (\tilde{\rho}(r_0), \tilde{\rho}(r_1), \tilde{u}(r_0)) = (\rho_0, \rho_1, j_0/\rho_0). \end{cases} \tag{1.5}$$

Obviously, each pair of the solution $(\tilde{\rho}, \tilde{u}, \tilde{E})$ for (1.5) always corresponds to a solution $(\rho, \mathbf{u}, E)$ of the system (1.2) with (1.4). The sonic state is redefined by $|\tilde{u}| = M = 1$, the flow of (1.5) is subsonic if $|\tilde{u}| < 1$, and the flow is supersonic if $|\tilde{u}| > 1$.

Let $\tilde{J} = \tilde{\rho}\tilde{u}$ be the current density. Without loss of generality, we also take $\tilde{J} > 0$. Integrating $(1.5)_1$ over $[r_0, r]$ and by $(1.5)_4$, we have

$$\tilde{J}(r) = \frac{j_0 r_0^{n-1}}{r^{n-1}}, \quad r \in [r_0, r_1], \tag{1.6}$$

and $(1.5)_2$ is reduced to

$$\left(1-\frac{\tilde{J}^2}{\tilde{\rho}^2}\right)\tilde{\rho}_r - \frac{n-1}{r}\frac{\tilde{J}^2}{\tilde{\rho}} = \tilde{\rho}\tilde{E} - \frac{\tilde{J}}{\tau}, \quad r \in (r_0, r_1). \tag{1.7}$$

Note that, the sonic state is $\tilde{u}=1$. Therefore, the sonic boundary value conditions of (1.5) are proposed as

$$\rho_0 = j_0, \qquad \rho_1 = \frac{j_0 r_0^{n-1}}{r_1^{n-1}}. \tag{1.8}$$

In order to further simplify the system (1.5), we define $\mathcal{J} := j_0 r_0^{n-1} > 0$ and introduce a new variable

$$m(r) := r^{n-1}\tilde{\rho}(r), \quad r \in [r_0, r_1].$$

It then follows from (1.6) that

$$\tilde{J} = \frac{\mathcal{J}}{r^{n-1}}, \qquad \tilde{\rho} = \frac{m}{r^{n-1}}, \qquad \tilde{u} = \frac{\mathcal{J}}{m}, \qquad r \in [r_0, r_1], \tag{1.9}$$

so that (1.5) with sonic boundary value conditions (1.8) are transformed into

$$\begin{cases} \left(1-\dfrac{\mathcal{J}^2}{m^2}\right)m_r = m\left(\tilde{E}+\dfrac{n-1}{r}\right) - \dfrac{\mathcal{J}}{\tau}, \\ (r^{n-1}\tilde{E})_r = m - B(r), \\ m(r_0) = m(r_1) = \mathcal{J}, \end{cases} \tag{1.10}$$

where the function $B(r)$ is defined by

$$B(r) := r^{n-1}\tilde{b}(r) \quad \text{for } r \in [r_0, r_1].$$

Throughout this paper, we denote

$$\underline{B} = \operatorname*{ess\,inf}_{r\in[r_0,r_1]} B(r), \qquad \overline{B} = \operatorname*{ess\,sup}_{r\in[r_0,r_1]} B(r).$$

As a result, to find a solution of (1.2), (1.4), and (1.8) is equivalent to solving (1.10). Clearly, the flow of the new system (1.10) is subsonic if $m > \mathcal{J}$ and is supersonic if $0 < m < \mathcal{J}$. Furthermore, $(1.10)_1$ is degenerate at the boundary, which will cause us some essential difficulties.

Background of research. The existence of subsonic/supersonic/transonic solutions to the stationary hydrodynamic model for semiconductors has been extensively investigated. For subsonic flows, Degond and Markowich [13] first showed the existence and uniqueness of subsonic solutions under a strongly subsonic background on a one-dimensional (1-d) bounded domain, and in [14], they further proved the existence and local uniqueness of smooth solutions under a smallness assumptions on the data in a three-dimensional bounded domain. Since then, the subsonic steady-states have been studied in different cases (see [1, 4, 5, 24]). For supersonic flows, Peng and Violet [29] showed the existence and uniqueness of supersonic solutions under a strongly supersonic background on a 1-D bounded domain. Regarding the transonic flows, when the boundary setting is still subsonic, but the doping profile is supersonic, by carrying out phase-plane analysis, Ascher et al. [2] first observed that there are infinitely many transonic steady-states with shocks for the system. This result was then generalized by Rosini [30] for the potential flows. On the other hand, by using the method of vanishing viscosity, Gamba technically constructed 1-D transonic steady-states with shocks in [17] and two-dimensional (2-D) transonic shocks in [18]. When the boundary data are separated by the sonic line, namely, one side boundary is supersonic, and the other side is subsonic, Luo and Xin [23] thoroughly studied the structure of stationary transonic shocks, in the case of relaxation time $\tau = \infty$, namely, no semiconductor effect related. Regarding the smooth transonic steady-states, Wang and Xin [34] technically studied the smooth transonic flows of Meyer type in de Laval nozzles, and Weng-Xin-Yuan [37] further proved the existence of smooth symmetric transonic flows for the nozzles with nonzero angular velocity and vorticity. Very recently, by carrying out analysis on the local singularity, Wei et al. [36] observed two different types of C^∞-smooth transonic steady-states.

When the boundary values are on the sonic line, the structure of the physical solutions becomes more complicated and sundry. Mei and his group [10, 11, 12, 20, 21, 36] first proposed this critical boundary case, and realized that the doping profile plays a crucial role for the existence/nonexistence of the physical solutions to the stationary system of semiconductor models. For the 1-D stationary Euler–Poisson system, when the doping profile is subsonic, Li et al. [20] proved that there exist many types of physical solutions: a unique interior subsonic solution, at least one interior supersonic solution, infinitely many transonic shocks once the relation time is large, and infinitely many C^1-smooth transonic solutions once the relation time is small. When the doping profile is supersonic, they [21] showed that the Euler–Poisson system with sonic boundary usually does not possess any physical solutions. Only in the case that the doping profile is sufficiently close to the sonic line, there exist a supersonic solution and infinitely many transonic shocks, and no subsonic solutions. When the doping profile is transonic, Chen et al. [10] further showed that the existence/nonexistence of these subsonic/supersonic/transonic solutions are dependent upon the domination of subsonic region or supersonic region of the doping profile. Precisely saying, when the subsonic region of doping profile is dominated, the structure of physical solutions is similar to the case of subsonic doping profile studied in [20], while, when the supersonic region of doping profile is dominated, then the structure of physical solutions is similar to the case of supersonic doping profile studied in [21]. The 2-D and three-dimensional (3-D) radial cases were further investigated in [11, 12]. The quasi-neutral limits were studied in [9].

Regarding the Euler–Poisson system without the semiconductor effect (the case of $\tau = \infty$) for nozzle flows, the subsonic/supersonic/transonic steady-states with the subsonic boundary/supersonic boundary/transonic boundary have been extensively studied. See [3, 4, 5, 6, 7, 32, 33, 35] and references therein. For the subsonic flows for the thermal semiconductor models, we refer to [26, 27, 28].

Motivation and difficulty. Note that the doping profile plays a key role for the existence/nonexistence of all types of physical solutions to the stationary hydrodynamic system with sonic boundary. So, from both of mathematical and physical points of view, it is quite interesting and important to study the structural stability of these steady-states, when the doping profiles are perturbed. Namely, once the perturbations of doping profiles are small, are the corresponding subsonic/supersonic/transonic solutions regarded also as small perturbations? This issue was first addressed by Luo et al. [22], where the transonic shocks are proved to be structurally stable in the case of $\tau = \infty$ (no semiconductor effect) with transonic boundary. Since these transonic shocks jump the sonic line without interaction, there is no singularity for the structural stability near the sonic line. Recently, Feng, Mei, and Zhang [16] attempted the case of C^1-smooth transonic steady-states, where the transonic solutions cross the sonic line, and the singularity for the system near the sonic line are formed. By taking some technical analysis on the local singularity, they [16] successfully proved the structural stability of the smooth transonic solutions once the doping profile is a small perturbation. When the boundary data are sonic, as showed in [20], the smooth subsonic/supersonic/transonic steady-states possess some serious singularities at the boundary, because the derivatives of these physical solutions at the sonic boundary are $-\infty$. For the 1-D case with subsonic doping profile, by using the technical weighted-energy method with the help of monotonicity argument, Feng, Hu, and -Mei [15] solved the structural stability of the subsonic steady-states. However, for the multiple dimensional case with sonic boundary, the structural stability is more challenging and significant to study. This is our study motivation in the present paper.

The main objective of this paper is to investigate the structural stability of interior subsonic solutions to the sonic boundary value problem (1.10) in 2-D and 3-D cases. There are some technical issues and difficulties in this work. First, we need to properly set down the multiple dimensional system with the sonic boundary. The other is to treat the challenging singularity of subsonic steady-states at the sonic boundary in the proof of structural stability. Owing to the boundary degeneracy, we need to analyze the boundary behavior of the first order derivative of $m_i(r)\,(i=1,2)$ at two endpoints, and to discover whether the singularity will occur at the two endpoints, which is important for us to establish the local structural stability estimates near the two endpoints. The optimal weight function will be introduced to treat the singularity of the derivatives of subsonic steady-states at the sonic boundary. The monotonicity argument will also play a crucial role in establishing the structural stability estimates near two endpoints. Besides, different from [15], we don't need to add the monotonicity restriction on doping profiles anymore through further analysis and discussion.

The rest of this paper is organized as follows. In section 2, we give the important preliminaries from the foregoing research, and then we state the main result of this paper. In section 3, we prove the structural stability of interior subsonic solutions to the sonic boundary value problem (1.10) in 2-D and 3-D cases.

2. Preliminaries and the main result. In this section, we first recall the existence and uniqueness of the interior subsonic solution to the sonic boundary value problem (1.10) in 2-D and 3-D cases, which have been obtained in [11]. Then, we state the main result of this paper.

Because the system (1.10) is degenerate at the boundary, we have to define the interior subsonic solution in the weak sense as that in [11, 20].

DEFINITION 2.1. *$m(r)$ is said to be an interior subsonic solution to the sonic boundary value problem* (1.10) *if we have the following:*

(i) $(m-\mathcal{J})^2\in H_0^1((r_0,r_1))$;

(ii) $m(r)>\mathcal{J}$ *for* $r\in(r_0,r_1)$;

(iii) $m(r_0)=m(r_1)=\mathcal{J}$;

(iv) *for any test function* $\varphi\in H_0^1((r_0,r_1))$, *it holds that*

$$\int_{r_0}^{r_1}\left(r^{n-1}\left(\frac{1}{m}-\frac{\mathcal{J}^2}{m^3}\right)m_r+\frac{r^{n-1}\mathcal{J}}{\tau m}\right)\varphi_r dr + \int_{r_0}^{r_1}(m-B(r)+r^{n-3}(n-1)(n-2))\varphi dr=0;$$

(v) $\tilde{E}(r)$ *is denoted by*

$$\tilde{E}(r) = \frac{r_0^{n-1}}{r^{n-1}}\tilde{E}(r_0) + \frac{1}{r^{n-1}}\int_{r_0}^{r}(m-B)(s)ds, \tag{2.1}$$

where $\tilde{E}(r_0) = \frac{1}{\tau} - \frac{n-1}{r_0}$.

We recall the existence and uniqueness of interior subsonic solution to the sonic boundary value problem (1.10) in 2-D and 3-D cases (Theorem 1.4 in [11]) as follows.

PROPOSITION 2.2 (existence of interior subsonic solutions [11]).

1. *Two-dimensional case:* $n = 2$. *Suppose that the doping profile* $B(r) \in L^\infty((r_0,r_1))$ *and* $\underline{B} \le B(r) \le \overline{B}$ *satisfying* $\overline{B} + \frac{1}{\tau} > \mathcal{J}$ *and* $\underline{B} + \frac{\mathcal{J}}{\tau(\overline{B}+1/\tau)} > \mathcal{J}$, *then the sonic boundary value problem* (1.10) *admits a unique interior subsonic solution* $(m,\tilde{E})(r) \in C^{1/2}([r_0,r_1]) \times H^1((r_0,r_1))$. *Moreover,* $m(r)$ *satisfies the estimate*

$$\mathcal{J} + \lambda \sin\left(\pi \cdot \frac{r-r_0}{r_1-r_0}\right) \le m(r) \le \overline{B} + \frac{1}{\tau}, \quad r \in [r_0,r_1], \tag{2.2}$$

where $\lambda > 0$ *is a small constant depending only on* r_0, r_1, $\mathcal{J}$, τ, $\underline{B}$, *and* $\overline{B}$.

2. *Three-dimensional case:* $n = 3$. *Define* $\underline{\mathcal{B}} := \inf_{r\in[r_0,r_1]}\left\{B(r) + \frac{2r}{\tau} - 2\right\}$ *and* $\overline{\mathcal{B}} := \sup_{r\in[r_0,r_1]}\left\{B(r) + \frac{2r}{\tau} - 2\right\}$. *Assume that* $\overline{\mathcal{B}} > \mathcal{J}$ *and* $\min_{r\in[r_0,r_1]}(B(r) + \frac{2r\mathcal{J}}{\tau\overline{\mathcal{B}}} - 2) > \mathcal{J}$, *then the sonic boundary value problem* (1.10) *possesses a unique interior subsonic solution* $(m,\tilde{E})(r) \in C^{1/2}([r_0,r_1]) \times H^1((r_0,r_1))$. *Moreover,* $m(r)$ *satisfies the estimate*

$$\mathcal{J} + \bar{\lambda} \sin\left(\pi \cdot \frac{r-r_0}{r_1-r_0}\right) \le m(r) \le \overline{\mathcal{B}}, \quad r \in [r_0,r_1], \tag{2.3}$$

where $\bar{\lambda} > 0$ *is a small constant depending only on* r_0, r_1, $\mathcal{J}$, τ, $\underline{\mathcal{B}}$, *and* $\overline{\mathcal{B}}$.

It is noticed that the sonic boundary value problem (1.10) is degenerate only at the boundary. By the standard theory for elliptic interior regularity and Sobolev's embedding theorem, if we assume the doping profile $B(r) \in C([r_0,r_1])$, then the corresponding interior subsonic solution $(m,\tilde{E})(r) \in \left(C^1((r_0,r_1)) \cap C^{1/2}([r_0,r_1])\right) \times C^1([r_0,r_1])$.

Now we state our main result for the structural stability of interior subsonic steady-states as follows.

THEOREM 2.3. *For* $i = 1,2$, *assume that doping profiles* $B_i(r) \in C([r_0,r_1])$, $\mathcal{J} < \underline{B}_i \le B_i(r) \le \overline{B}_i$ *on* $[r_0,r_1]$ *for* $n = 2$, *and* $\mathcal{J} + 2 < \underline{B}_i \le B_i(r) \le \overline{B}_i$ *on* $[r_0,r_1]$ *for* $n = 3$. *Let* $(m_i,\tilde{E}_i)(r)$ *be two interior subsonic solutions to the sonic boundary value problem* (1.10) *corresponding to doping profiles* $B_i(r)$. *Then there exists a constant* $\tau_0(r_0,B_1(r_0),B_2(r_0))$ *such that for any* $0 < \tau < \tau_0$, *the interior subsonic solutions* $(m_i,\tilde{E}_i)(r)$ *to* (1.10) *are structurally stable, namely,*

$$\begin{aligned}&\|m_1 - m_2\|_{C([r_0,r_1])} + \|(r_1-r)^{1/2}(m_1-m_2)_r\|_{C([r_0,r_1])}\\ &\quad + \|\tilde{E}_1 - \tilde{E}_2\|_{C^1([r_0,r_1])} < C\|B_1 - B_2\|_{C([r_0,r_1])},\end{aligned}$$

where C *is a positive constant independent of* $\|B_1 - B_2\|_{C([r_0,r_1])}$.

Because the interior subsonic solution of (1.10) is degenerate at the boundary, it is difficult to directly study the structural stability of solutions on the whole domain

$[r_0,r_1]$. Therefore, we have to divide the whole domain $[r_0,r_1]$ into three domains as below:

$$[r_0,r_1]=[r_0,r_0+\delta_0)\cup[r_0+\delta_0,r_1-\delta_0]\cup(r_1-\delta_0,r_1],$$

where $\delta_0\in(0,(r_1-r_0)/2)$ would be determined later (see Lemma 3.8 for details). And then we will establish the structural stability estimates on three domains, respectively. The most important thing is that we need to analyze the boundary behavior of the first order derivative of $m_i(r)\,(i=1,2)$ at two endpoints, and confirm whether the two endpoints are singular points. The monotonicity argument will also play a crucial role in establishing the structural stability estimates near two endpoints.

3. The structural stability of interior subsonic solutions. In this section, we are going to prove the structural stability of interior subsonic solutions to the sonic boundary value problem (1.10) for both 2-D and 3-D cases, in other words, we devote ourselves to proving Theorem 2.3.

We first focus on proving the structural stability of interior subsonic solutions to the sonic boundary value problem (1.10) in the 2-D case. For $i=1,2$, let $(m_i,\tilde{E}_i)(r)$ denote the interior subsonic solutions to (1.10) with $n=2$ relative to doping profiles $B_i(r)$, then $(m_i,\tilde{E}_i)(r)$ satisfy the following system:

$$\begin{cases}\left(1-\dfrac{\mathcal{J}^2}{(m_i)^2}\right)(m_i)_r=m_i\left(\tilde{E}_i+\dfrac{1}{r}\right)-\dfrac{\mathcal{J}}{\tau}, & \\ (r\tilde{E}_i)_r=m_i-B_i(r), & r\in(r_0,r_1), \\ m_i(r_0)=m_i(r_1)=\mathcal{J}. & \end{cases} \tag{3.1}$$

THEOREM 3.1. *For $i=1,2$, let doping profiles $B_i(r)\in C\big([r_0,r_1]\big)$, $\mathcal{J}<\underline{B}_i\le B_i(r)\le\overline{B}_i$ on $[r_0,r_1]$, and τ satisfy*

$$0<\tau<\min\left\{\frac{-1+\sqrt{1+r_0(B_1(r_0)-\mathcal{J})/2}}{2(B_1(r_0)-\mathcal{J})},\ \frac{-1+\sqrt{1+r_0(B_2(r_0)-\mathcal{J})/2}}{2(B_2(r_0)-\mathcal{J})},\ r_0\right\}. \tag{3.2}$$

Then, the interior subsonic solutions $(m_i,\tilde{E}_i)(r)$ to (3.1) *are structurally stable in the sense that*

$$\begin{aligned}&\|m_1-m_2\|_{C([r_0,r_1])}+\|(r_1-r)^{1/2}(m_1-m_2)_r\|_{C([r_0,r_1])}\\&\quad+\|\tilde{E}_1-\tilde{E}_2\|_{C^1([r_0,r_1])}<C\|B_1-B_2\|_{C([r_0,r_1])},\end{aligned} \tag{3.3}$$

where $C>0$ is a constant independent of $\|B_1-B_2\|_{C([r_0,r_1])}$.

In order to use the monotonicity argument to establish the structural stability estimates near two endpoints, we introduce the following modified comparison principle.

LEMMA 3.2 (comparison principle). *Suppose that the conditions in Theorem* 3.1 *hold. If $B_1(r)\ge B_2(r)$ on $[r_0,r_1]$, then,*

$$m_1(r)\ge m_2(r)\quad for\quad r\in[r_0,r_1]. \tag{3.4}$$

Proof. For $i=1,2$, substituting $(3.1)_2$ into $(3.1)_1$ gives

$$\begin{cases} \left(r\left(\frac{1}{m_i}-\frac{\mathcal{J}^2}{(m_i)^3}\right)(m_i)_r\right)_r+\left(\frac{r\mathcal{J}}{\tau m_i}\right)_r-(m_i-B_i(r))=0, & r\in(r_0,r_1),\\ m_i(r_0)=m_i(r_1)=\mathcal{J}. \end{cases} \tag{3.5}$$

Since m_i is the interior subsonic solution to (3.5), we have the approximate solution sequence $\{m_{ij}\}_{0<j<\mathcal{J}}\subset C^1([r_0,r_1])$ which satisfies the following integral equality:

$$\begin{aligned}\int_{r_0}^{r_1} r\left[\left(\frac{1}{m_{ij}}-\frac{j^2}{(m_{ij})^3}\right)(m_{ij})_r+\frac{j}{\tau m_{ij}}\right]\varphi_r dr\\ +\int_{r_0}^{r_1}(m_{ij}-B_i)\varphi dr=0 \quad \text{for any } \varphi\in H_0^1((r_0,r_1)).\end{aligned} \tag{3.6}$$

Let us denote

$$A(U,V)=\left(\frac{1}{U}-\frac{j^2}{U^3}\right)V+\frac{j}{\tau U}.$$

Subtracting (3.6) with $i=1$ from (3.6) with $i=2$, for any $\varphi\geq 0$ and $\varphi\in H_0^1((r_0,r_1))$, by $B_1(r)\geq B_2(r)$ on $[r_0,r_1]$, we obtain

$$\begin{aligned}\int_{r_0}^{r_1} r[A(m_{2j},(m_{2j})_r)-A(m_{1j},(m_{1j})_r)]\varphi_r dr+\int_{r_0}^{r_1}(m_{2j}-m_{1j})\varphi dr\\ =\int_{r_0}^{r_1}(B_2-B_1)\varphi dr\leq 0.\end{aligned} \tag{3.7}$$

Inequality (3.7) is similar to (19) in Lemma 2.2 of [20], owing to $r\in[r_0,r_1]$, if we apply the same arguments as that in Lemma 2.2 of [20], we have the same result as Lemma 2.2 of [20], namely,

$$m_{1j}(r)\geq m_{2j}(r) \quad \text{for} \quad r\in[r_0,r_1], \quad 0<j<\mathcal{J}. \tag{3.8}$$

Taking the limit as $j\to\mathcal{J}^-$ on both side of (3.8), we can get (3.4). ◻

First, we establish the local structural stability estimate of interior subsonic solutions near the left endpoint $r=r_0$. Before we establish the local structural stability estimate, we need to analyze the boundary behavior of the first order derivative of $m_i(r)\,(i=1,2)$ at $r=r_0$. Owing to $\tilde{E}_i(r_0)=\frac{1}{\tau}-\frac{1}{r_0}$, the plausible singularity at the left endpoint $r=r_0$ could be removable.

LEMMA 3.3. *Assume that the conditions in Theorem* 3.1 *are satisfied. Then,*

$$\lim_{r\to r_0^+}(m_i)_r(r)=\frac{\mathcal{J}}{4}\left(\frac{1}{\tau}-\sqrt{\frac{1}{\tau^2}-\frac{8}{r_0\tau}(1+(B_i(r_0)-\mathcal{J})\tau)}\right)=:A_i>0, \quad i=1,2. \tag{3.9}$$

Proof. Thanks to $m_i(r_0)=\mathcal{J}$ and $\tilde{E}_i(r_0)=\frac{1}{\tau}-\frac{1}{r_0}>0$, we know that $\lim_{r\to r_0^+}(m_i)_r(r)$ exists, which has been proved in Theorem 3.6 of [12].

By $(3.1)_1$, we have

$$(m_i)_r=\frac{(m_i)^2\tilde{E}_i}{m_i+\mathcal{J}}+\frac{(m_i)^2}{(m_i+\mathcal{J})(m_i-\mathcal{J})}\left(\mathcal{J}\tilde{E}_i-\left(\frac{\mathcal{J}}{\tau}-\frac{m_i}{r}\right)\right).$$

Applying the L'Hospital rule and $(3.1)_2$, we calculate that

$$\begin{aligned}
A_i &= \lim_{r\to r_0^+}(m_i)_r \\
&= \lim_{r\to r_0^+}\frac{(m_i)^2\tilde{E}_i}{m_i+\mathcal{J}} + \lim_{r\to r_0^+}\frac{(m_i)^2}{(m_i+\mathcal{J})(m_i-\mathcal{J})}\left(\mathcal{J}\tilde{E}_i - \left(\frac{\mathcal{J}}{\tau}-\frac{m_i}{r}\right)\right) \\
&= \frac{\mathcal{J}}{2}\left(\frac{1}{\tau}-\frac{1}{r_0}\right) + \frac{\mathcal{J}}{2}\lim_{r\to r_0^+}\frac{1}{m_i-\mathcal{J}}\left(\mathcal{J}\tilde{E}_i - \left(\frac{\mathcal{J}}{\tau}-\frac{m_i}{r}\right)\right) \\
&= \frac{\mathcal{J}}{2}\left(\frac{1}{\tau}-\frac{1}{r_0}\right) + \frac{\mathcal{J}}{2}\lim_{r\to r_0^+}\frac{1}{(m_i)_r}\left(\mathcal{J}(\tilde{E}_i)_r + \frac{(m_i)_r}{r} - \frac{m_i}{r^2}\right) \\
&= \frac{\mathcal{J}}{2}\left(\frac{1}{\tau}-\frac{1}{r_0}\right) + \frac{\mathcal{J}^2}{2}\lim_{r\to r_0^+}\frac{m_i-B_i-\tilde{E}_i}{r(m_i)_r} + \frac{\mathcal{J}}{2}\frac{1}{r_0} - \frac{\mathcal{J}}{2}\lim_{r\to r_0^+}\frac{m_i}{r^2(m_i)_r} \\
&= \frac{\mathcal{J}}{2\tau} + \frac{\mathcal{J}^2}{2}\frac{\mathcal{J}-B_i(r_0)}{r_0A_i} - \frac{\mathcal{J}^2}{2}\left(\frac{1}{\tau}-\frac{1}{r_0}\right)\frac{1}{r_0A_i} - \frac{\mathcal{J}^2}{2r_0^2A_i} \\
&= \frac{\mathcal{J}}{2\tau} + \frac{\mathcal{J}^2}{2r_0A_i}\left(\mathcal{J}-B_i(r_0)-\frac{1}{\tau}\right).
\end{aligned}$$

A direct computation indicates that there are two solutions:

$$A_i^{(1)} = \frac{\mathcal{J}}{4}\left(\frac{1}{\tau}+\sqrt{\frac{1}{\tau^2}-\frac{8}{r_0\tau}(1+(B_i(r_0)-\mathcal{J})\tau)}\right) = O\left(\frac{1}{\tau}\right)$$

or

$$A_i^{(2)} = \frac{\mathcal{J}}{4}\left(\frac{1}{\tau}-\sqrt{\frac{1}{\tau^2}-\frac{8}{r_0\tau}(1+(B_i(r_0)-\mathcal{J})\tau)}\right) = C\mathcal{J}+O(\tau).$$

According to the relevant argument in Theorem 3.6 of [12], we can exclude $A_i^{(1)}$, thus, we get (3.9). ∎

We are able to establish the local structural stability estimate of interior subsonic solutions to the boundary value problem (3.1) on an intrinsic neighborhood of the left endpoint $r=r_0$ by using Lemmas 3.2 and 3.3.

LEMMA 3.4. *Let all assumptions in Theorem* 3.1 *hold, and let* $B_1(r)\geq B_2(r)$ *on* $[r_0,r_1]$. *Then there are two positive constants* $\delta_1\in(0,(r_1-r_0)/2)$ *and* C *independent of* $\|B_1-B_2\|_{C([r_0,r_1])}$ *such that*

$$\|m_1-m_2\|_{C^1([r_0,r_0+\delta_1))} + \|\tilde{E}_1-\tilde{E}_2\|_{C^1([r_0,r_0+\delta_1))} \leq C\|B_1-B_2\|_{C([r_0,r_1])}. \tag{3.10}$$

Proof. Note that the function $g(s)=\frac{s^3}{s+\mathcal{J}}$ is increasing for $s>0$, thus, from Lemma 3.2, we have

$$\frac{(m_1)^3}{m_1+\mathcal{J}} \geq \frac{(m_2)^3}{m_2+\mathcal{J}}, \quad r\in[r_0,r_1]. \tag{3.11}$$

And it follows from (2.2) and (3.4) that

$$\mathcal{J}\leq\mathcal{J}+\lambda_2\sin\left(\pi\cdot\frac{r-r_0}{r_1-r_0}\right)\leq m_2(r)\leq m_1(r)\leq\overline{B}_1+\frac{1}{\tau}, \quad r\in[r_0,r_1]. \tag{3.12}$$

By $(3.1)_1$, we get, for $i=1,2$,

$$(m_i)_r = \frac{(m_i)^3}{(m_i)^2-\mathcal{J}^2}\left(\tilde{E}_i+\frac{1}{r}-\frac{\mathcal{J}}{m_i\tau}\right) =: \frac{(m_i)^3\hat{E}_i}{(m_i)^2-\mathcal{J}^2}. \tag{3.13}$$

Taking the difference of $(3.13)|_{i=1}$ and $(3.13)|_{i=2}$, one has

$$
\begin{aligned}
(3.14) \qquad &(m_1-m_2)_r\\
&=\frac{(m_1)^3\hat{E}_1}{(m_1)^2-\mathcal{J}^2}-\frac{(m_2)^3\hat{E}_2}{(m_2)^2-\mathcal{J}^2}\\
&=\frac{(m_1)^3}{m_1+\mathcal{J}}\frac{\hat{E}_1}{m_1-\mathcal{J}}-\frac{(m_2)^3}{m_2+\mathcal{J}}\frac{\hat{E}_1}{m_1-\mathcal{J}}\\
&\quad+\frac{(m_2)^3}{m_2+\mathcal{J}}\frac{\hat{E}_1}{m_1-\mathcal{J}}-\frac{(m_2)^3}{m_2+\mathcal{J}}\frac{\hat{E}_2}{m_2-\mathcal{J}}\\
&=\frac{\hat{E}_1}{m_1-\mathcal{J}}\left(\frac{(m_1)^3}{m_1+\mathcal{J}}-\frac{(m_2)^3}{m_2+\mathcal{J}}\right)+\frac{(m_2)^3}{m_2+\mathcal{J}}\left(\frac{\hat{E}_1}{m_1-\mathcal{J}}-\frac{\hat{E}_2}{m_2-\mathcal{J}}\right).
\end{aligned}
$$

We claim that there exist two positive constants $\delta_1\in(0,(r_1-r_0)/2)$ and M_1 independent of $\|B_1-B_2\|_{C([r_0,r_1])}$ such that

$$
\begin{aligned}
(3.15) \qquad &\frac{\hat{E}_1}{m_1-\mathcal{J}}(r)\le\frac{M_1}{\mathcal{J}\tau}, \qquad r\in[r_0,r_0+\delta_1), \quad \text{and}\\
&\left(\frac{\hat{E}_1}{m_1-\mathcal{J}}-\frac{\hat{E}_2}{m_2-\mathcal{J}}\right)(r)\le M_1\|B_1-B_2\|_{C([r_0,r_1])}, \quad r\in[r_0,r_0+\delta_1).
\end{aligned}
$$

Therefore, using the monotonicity relation (3.11) and substituting (3.15) to (3.14), we obtain

$$
\begin{aligned}
(3.16) \qquad (m_1-m_2)_r&\le\frac{M_1}{\mathcal{J}\tau}\left(\frac{(m_1)^3}{m_1+\mathcal{J}}-\frac{(m_2)^3}{m_2+\mathcal{J}}\right)+M_1\|B_1-B_2\|_{C([r_0,r_1])}\frac{(m_2)^3}{m_2+\mathcal{J}}\\
&\le C(m_1-m_2)+C\|B_1-B_2\|_{C([r_0,r_1])}, \quad r\in[r_0,r_0+\delta_1),
\end{aligned}
$$

where we used the mean-value theorem of differentials and (3.12) in the last inequality.

Now we prove that the estimates in (3.15) hold. Otherwise, for any $\delta\in(0,(r_1-r_0)/2)$ and $M>0$, there exists $r_\delta\in[r_0,r_0+\delta)$ such that

$$
(3.17) \qquad \frac{\hat{E}_1}{m_1-\mathcal{J}}(r_\delta)>\frac{M}{\mathcal{J}\tau} \quad \text{or} \quad \left(\frac{\hat{E}_1}{m_1-\mathcal{J}}-\frac{\hat{E}_2}{m_2-\mathcal{J}}\right)(r_\delta)>M\|B_1-B_2\|_{C([r_0,r_1])}.
$$

Due to the arbitrariness of δ, we choose $\delta=(r_1-r_0)/n$ with $n=3,4,5,\dots$, then, for any $M>0$, there exists $r_n\in[r_0,r_0+\delta)$ such that

$$
\frac{\hat{E}_1}{m_1-\mathcal{J}}(r_n)>\frac{M}{\mathcal{J}\tau} \quad \text{or} \quad \left(\frac{\hat{E}_1}{m_1-\mathcal{J}}-\frac{\hat{E}_2}{m_2-\mathcal{J}}\right)(r_n)>M\|B_1-B_2\|_{C([r_0,r_1])},
$$

which indicate that

$$
(3.18) \qquad \varliminf_{r_n\to r_0^+}\frac{\hat{E}_1}{m_1-\mathcal{J}}(r_n)\ge\frac{M}{\mathcal{J}\tau}
$$

or

$$
(3.19) \qquad \varliminf_{r_n\to r_0^+}\left(\frac{\hat{E}_1}{m_1-\mathcal{J}}-\frac{\hat{E}_2}{m_2-\mathcal{J}}\right)(r_n)\ge M\|B_1-B_2\|_{C([r_0,r_1])}.
$$

By using the L'Hospital rule, $(3.1)_2$ and (3.9), we can derive, for $i=1,2$,

$$
\begin{aligned}
(3.20)\qquad &\lim_{r\to r_0^+}\frac{\hat{E}_i}{m_i-\mathcal{J}}(r)\\
&=\lim_{r\to r_0^+}\frac{1}{m_i-\mathcal{J}}\left(\tilde{E}_i+\frac{1}{r}-\frac{\mathcal{J}}{m_i\tau}\right)(r)\\
&=\lim_{r\to r_0^+}\frac{1}{(m_i)_r}\left((\tilde{E}_i)_r-\frac{1}{r^2}+\frac{\mathcal{J}(m_i)_r}{(m_i)^2\tau}\right)(r)\\
&=\lim_{r\to r_0^+}\frac{1}{(m_i)_r}\left(\frac{1}{r}(m_i-B_i-\tilde{E}_i)-\frac{1}{r^2}\right)(r)+\lim_{r\to r_0^+}\frac{\mathcal{J}}{(m_i)^2(r)\tau}\\
&=\frac{1}{A_i}\left(\frac{1}{r_0}\left(\mathcal{J}-B_i(r_0)-\left(\frac{1}{\tau}-\frac{1}{r_0}\right)\right)-\frac{1}{r_0^2}\right)+\frac{1}{\mathcal{J}\tau}\\
&=\frac{1}{r_0A_i}\left(\mathcal{J}-B_i(r_0)-\frac{1}{\tau}\right)+\frac{1}{\mathcal{J}\tau}<\frac{1}{\mathcal{J}\tau},
\end{aligned}
$$

and further have

$$
\begin{aligned}
(3.21)\qquad &\lim_{r\to r_0^+}\left(\frac{\hat{E}_1}{m_1-\mathcal{J}}-\frac{\hat{E}_2}{m_2-\mathcal{J}}\right)(r)\\
&=\frac{1}{r_0A_1}\left(\mathcal{J}-B_1(r_0)-\frac{1}{\tau}\right)-\frac{1}{r_0A_2}\left(\mathcal{J}-B_2(r_0)-\frac{1}{\tau}\right)\\
&=\frac{1}{r_0\tau}\left(\frac{\tau(\mathcal{J}-B_1(r_0))-1}{A_1}-\frac{\tau(\mathcal{J}-B_2(r_0))-1}{A_2}\right)\\
&=\frac{1}{2\mathcal{J}\tau}\left(\sqrt{1-\frac{8}{r_0}(\tau+(B_2(r_0)-\mathcal{J})\tau^2)}-\sqrt{1-\frac{8}{r_0}(\tau+(B_1(r_0)-\mathcal{J})\tau^2)}\right)\\
&=\frac{2\tau}{\mathcal{J}r_0}\frac{1}{\sqrt{1-\frac{8}{r_0}(\tau+(\xi-\mathcal{J})\tau^2)}}(B_1(r_0)-B_2(r_0))\\
&\le C_1\|B_1-B_2\|_{C([r_0,r_1])},
\end{aligned}
$$

where $\xi\in(B_2(r_0),B_1(r_0))$. Owing to the arbitrariness of M in (3.18) and (3.19), we choose $M=2$ in (3.18), which gives a contradiction to (3.20), and we choose $M=2C_1$ in (3.19), which contradicts (3.21).

Next, we multiply (3.16) by $(m_1-m_2)(r)$ and apply Cauchy's inequality to get

$$((m_1-m_2)^2)_r(r)\le C(m_1-m_2)^2(r)+C\|B_1-B_2\|^2_{C([r_0,r_1])},\quad r\in[r_0,r_0+\delta_1).$$

Applying Gronwall's inequality to above inequality gives

$$(m_1-m_2)^2(r)\le C\|B_1-B_2\|^2_{C([r_0,r_1])},\quad r\in[r_0,r_0+\delta_1),$$

thus,

$$(m_1-m_2)(r)\le C\|B_1-B_2\|_{C([r_0,r_1])},\quad r\in[r_0,r_0+\delta_1).$$

The above inequality together with (3.14) implies

$$(3.22)\qquad |m_1-m_2|(r)+|(m_1-m_2)_r|(r)\le C\|B_1-B_2\|_{C([r_0,r_1])},\quad r\in[r_0,r_0+\delta_1).$$

Finally, integrating $(3.1)_2$ over $[r_0, r]$ and recalling that $\tilde{E}_i(r_0) = \frac{1}{\tau} - \frac{1}{r_0}$, we obtain

$$\tilde{E}_i(r) = \frac{r_0}{r}\left(\frac{1}{\tau} - \frac{1}{r_0}\right) + \frac{1}{r}\int_{r_0}^{r} (m_i - B_i)(s)ds, \tag{3.23}$$

and further derive

$$\begin{aligned} |\tilde{E}_1 - \tilde{E}_2|(r) &\leq \frac{1}{r}\int_{r_0}^{r} |m_1 - m_2|(s)ds + \frac{1}{r}\int_{r_0}^{r} |B_1 - B_2|(s)ds \\ &\leq C\|B_1 - B_2\|_{C([r_0,r_1])}, \qquad r \in [r_0, r_0 + \delta_1). \end{aligned} \tag{3.24}$$

From $(3.1)_2$, we calculate that

$$(\tilde{E}_1 - \tilde{E}_2)_r(r) = -\frac{1}{r}(\tilde{E}_1 - \tilde{E}_2)(r) + \frac{1}{r}(m_1 - m_2)(r) - \frac{1}{r}(B_1 - B_2)(r). \tag{3.25}$$

It then follows from (3.22) and (3.24) that

$$|(\tilde{E}_1 - \tilde{E}_2)_r|(r) \leq C\|B_1 - B_2\|_{C([r_0,r_1])}, \quad r \in [r_0, r_0 + \delta_1). \tag{3.26}$$

Hence, the desired estimate (3.10) can be directly obtained from (3.22), (3.24), and (3.26). ∎

Second, we establish the local structural stability estimate of interior subsonic solutions near the right endpoint $r = r_1$. Before proceeding, we need to analyze the boundary behavior of the first order derivative of $m_i(r)\,(i = 1, 2)$ at $r = r_1$. Inspired by Proposition 2.5 in [20], we can derive the following estimates of interior subsonic solutions near $r = r_1$.

Lemma 3.5. *Let $B_i(r) \in L^\infty((r_0, r_1))$, $\underline{B}_i > \mathcal{J}\,(i = 1, 2)$, and $0 < \tau < r_0$. Then, $\tilde{E}_i(r_1) < \frac{1}{\tau} - \frac{1}{r_1}$, and there exist positive constants $\hat{C}_l\,(l = 1, 2, 3, 4)$ such that, for r near r_1,*

$$\hat{C}_1(r_1 - r)^{1/2} < m_i(r) - \mathcal{J} < \hat{C}_2(r_1 - r)^{1/2}, \tag{3.27}$$

$$-\hat{C}_3(r_1 - r)^{-1/2} < (m_i)_r(r) < -\hat{C}_4(r_1 - r)^{-1/2}, \tag{3.28}$$

where $\hat{C}_2 > \hat{C}_1$ and $\hat{C}_3 > \hat{C}_4$.

The proof of Lemma 3.5 is similar to that in Proposition 2.5 of [20] and is omitted here. It is easy to see that $\lim_{r \to r_1^-}(m_i)_r(r) = -\infty$ from (3.28), and then we deduce that the singularity will occur at the right endpoint $r = r_1$. But from (3.28), we know that the singularity at $r = r_1$ can be well controlled by the $(r_1 - r)^{1/2}$-weight, thus, we can establish the local weighted structural stability estimate near $r = r_1$.

Lemma 3.6. *Under the conditions of Theorem* 3.1*, it holds that*

$$\lim_{r \to r_1^-}(r_1 - r)^{1/2}(m_i)_r(r) = -\frac{\mathcal{J}}{2}\sqrt{\frac{1}{r_1}\int_{r_0}^{r_1}\left(B_i - m_i + \frac{1}{\tau}\right)(r)dr} =: a_i < 0, \quad i = 1, 2. \tag{3.29}$$

Proof. By (3.28), we can know that the coefficient $1 - \frac{\mathcal{J}^2}{(m_i)^2}$ in the degenerate principal part of $(3.1)_1$ is comparable to $(r_1 - r)^{1/2}$ near the endpoint $r = r_1$. Therefore, the regularity theory of boundary degenerate elliptic equations in 1-D case [31] ensures that $(r_1 - r)^{1/2}(m_i)_r(r)$ can be continuous up to the right endpoint $r = r_1$.

Now we compute $\lim_{r\to r_1^-}(r_1-r)^{1/2}(m_i)_r(r)$. Multiplying (3.13) by $(r_1-r)^{1/2}$, we have

$$(r_1-r)^{1/2}(m_i)_r=\frac{(m_i)^3}{m_i+\mathcal{J}}\left(\tilde{E}_i+\frac{1}{r}-\frac{\mathcal{J}}{\tau m_i}\right)\frac{(r_1-r)^{1/2}}{m_i-\mathcal{J}}. \tag{3.30}$$

From (3.30) and the L'Hospital rule, we calculate that

$$\begin{aligned} a_i &= \lim_{r\to r_1^-}(r_1-r)^{1/2}(m_i)_r \\ &= \lim_{r\to r_1^-}\frac{(m_i)^3}{m_i+\mathcal{J}}\lim_{r\to r_1^-}\left(\tilde{E}_i+\frac{1}{r}-\frac{\mathcal{J}}{\tau m_i}\right)\lim_{r\to r_1^-}\frac{(r_1-r)^{1/2}}{m_i-\mathcal{J}} \\ &= \frac{\mathcal{J}^2}{2}\left(\tilde{E}_i(r_1)+\frac{1}{r_1}-\frac{1}{\tau}\right)\lim_{r\to r_1^-}\frac{-\frac{1}{2}(r_1-r)^{-1/2}}{(m_i)_r} \\ &= \frac{\mathcal{J}^2}{4}\left(\tilde{E}_i(r_0)-\tilde{E}_i(r_1)+\frac{1}{r_0}-\frac{1}{r_1}\right)\frac{1}{a_i}. \end{aligned}$$

It follows from (3.28) that we only reserve a negative root:

$$\begin{aligned} a_i &= -\frac{\mathcal{J}}{2}\sqrt{\tilde{E}_i(r_0)-\tilde{E}_i(r_1)+\frac{1}{r_0}-\frac{1}{r_1}} \\ &= -\frac{\mathcal{J}}{2}\sqrt{\frac{1}{r_1}\int_{r_0}^{r_1}\left(B_i-m_i+\frac{1}{\tau}\right)(r)dr}, \end{aligned}$$

where we used (3.23) in the last equality. ∎

We are able to establish the local weighted structural stability estimate of interior subsonic solutions to the sonic boundary value problem (3.1) on an intrinsic neighborhood of the right endpoint $r=r_1$ by applying Lemma 3.2 and Lemmas 3.5 and Lemma 3.6.

LEMMA 3.7. *Let all assumptions in Theorem* 3.1 *be satisfied, and let* $B_1(r)\geq B_2(r)$ *on* $[r_0,r_1]$. *Then there exist two positive constants* $\delta_2\in(0,(r_1-r_0)/2)$ *and* C *independent of* $\|B_1-B_2\|_{C([r_0,r_1])}$ *such that*

$$\begin{aligned} &\|(r_1-r)^{-1/2}(m_1-m_2)\|_{C((r_1-\delta_2,r_1])}+\|(r_1-r)^{1/2}(m_1-m_2)_r\|_{C((r_1-\delta_2,r_1])} \\ &\quad+\|\tilde{E}_1-\tilde{E}_2\|_{C^1((r_1-\delta_2,r_1])}<C\|B_1-B_2\|_{C([r_0,r_1])}. \end{aligned} \tag{3.31}$$

Proof. It follows from the estimate (3.27) that $\frac{m_i(r)-\mathcal{J}}{(r_1-r)^{1/2}}$ $(i=1,2)$ has the uniform positive lower and upper bounds near $r=r_1$, and then its reciprocal also has this property.

Owing to the singularity occurs at $r=r_1$, we can only establish the weighted structural stability estimate near the right endpoint $r=r_1$.

Taking the difference of $(3.30)|_{i=1}$ and $(3.30)|_{i=2}$, we get

$$\begin{aligned} &(r_1-r)^{1/2}(m_1-m_2)_r \\ &=\frac{(m_1)^2}{m_1+\mathcal{J}}\left(m_1\left(\tilde{E}_1+\frac{1}{r}\right)-\frac{\mathcal{J}}{\tau}\right)\frac{(r_1-r)^{1/2}}{m_1-\mathcal{J}} \\ &\quad-\frac{(m_2)^2}{m_2+\mathcal{J}}\left(m_2\left(\tilde{E}_2+\frac{1}{r}\right)-\frac{\mathcal{J}}{\tau}\right)\frac{(r_1-r)^{1/2}}{m_2-\mathcal{J}} \end{aligned} \tag{3.32}$$

$$
\begin{aligned}
&= \frac{(m_1)^2}{m_1+\mathcal{J}}\left(m_1\left(\tilde{E}_1+\frac{1}{r}\right)-\frac{\mathcal{J}}{\tau}\right)\left(\frac{(r_1-r)^{1/2}}{m_1-\mathcal{J}}-\frac{(r_1-r)^{1/2}}{m_2-\mathcal{J}}\right)\\
&\quad+\left[\frac{(m_1)^2}{m_1+\mathcal{J}}\left(m_1\left(\tilde{E}_1+\frac{1}{r}\right)-\frac{\mathcal{J}}{\tau}\right)\right.\\
&\quad\left.-\frac{(m_2)^2}{m_2+\mathcal{J}}\left(m_2\left(\tilde{E}_2+\frac{1}{r}\right)-\frac{\mathcal{J}}{\tau}\right)\right]\frac{(r_1-r)^{1/2}}{m_2-\mathcal{J}}\\
&= f(m_1,\tilde{E}_1)\frac{(r_1-r)^{1/2}}{m_1-\mathcal{J}}\frac{(r_1-r)^{1/2}}{m_2-\mathcal{J}}\frac{m_2-m_1}{(r_1-r)^{1/2}}\\
&\quad+(f(m_1,\tilde{E}_1)-f(m_2,\tilde{E}_2))\frac{(r_1-r)^{1/2}}{m_2-\mathcal{J}}\\
&=: I_1+I_2,
\end{aligned}
$$

where

$$
f(m_i,\tilde{E}_i)=\frac{(m_i)^2}{m_i+\mathcal{J}}\left(m_i\left(\tilde{E}_i+\frac{1}{r}\right)-\frac{\mathcal{J}}{\tau}\right),\quad i=1,2.
$$

Before we estimate I_1 and I_2 near $r=r_1$, we first claim that

$$
|\tilde{E}_i(r)|<C(\overline{B}_1+1/\tau),\quad r\in[r_0,r_1],\quad i=1,2, \tag{3.33}
$$

$$
|\tilde{E}_1(r_1)-\tilde{E}_2(r_1)|<C\|B_1-B_2\|_{C([r_0,r_1])}, \tag{3.34}
$$

where the positive constant C is independent of $\|B_1-B_2\|_{C([r_0,r_1])}$. The estimates (3.33)–(3.34) will be verified in Lemma 3.9.

Now we estimate I_1 and I_2 near $r=r_1$, respectively. It follows from (3.12), (3.27), and (3.33) that I_1 can be estimated as

$$
|I_1|=\left|f(m_1,\tilde{E}_1)\frac{(r_1-r)^{1/2}}{m_1-\mathcal{J}}\frac{(r_1-r)^{1/2}}{m_2-\mathcal{J}}\frac{m_2-m_1}{(r_1-r)^{1/2}}\right|<C\frac{|m_1-m_2|}{(r_1-r)^{1/2}}. \tag{3.35}
$$

We estimate I_2 as below

$$
\begin{aligned}
|I_2|&=\left|(f(m_1,\tilde{E}_1)-f(m_2,\tilde{E}_2))\frac{(r_1-r)^{1/2}}{m_2-\mathcal{J}}\right|\\
&<C|f(m_1,\tilde{E}_1)-f(m_2,\tilde{E}_2)|\\
&=C\left|\frac{1}{r}\left(\frac{(m_1)^3}{m_1+\mathcal{J}}-\frac{(m_2)^3}{m_2+\mathcal{J}}\right)-\frac{\mathcal{J}}{\tau}\left(\frac{(m_1)^2}{m_1+\mathcal{J}}-\frac{(m_2)^2}{m_2+\mathcal{J}}\right)\right.\\
&\quad\left.+\frac{(m_1)^3}{m_1+\mathcal{J}}\tilde{E}_1-\frac{(m_2)^3}{m_2+\mathcal{J}}\tilde{E}_2\right|\\
&<C|m_1-m_2|+C\left|\left(\frac{(m_1)^3}{m_1+\mathcal{J}}-\frac{(m_2)^3}{m_2+\mathcal{J}}\right)\tilde{E}_1+\frac{(m_2)^3}{m_2+\mathcal{J}}(\tilde{E}_1-\tilde{E}_2)\right|\\
&<C|m_1-m_2|+C|\tilde{E}_1-\tilde{E}_2|\\
&<C\frac{|m_1-m_2|}{(r_1-r)^{1/2}}+C\left|\frac{r_1}{r}(\tilde{E}_1(r_1)-\tilde{E}_2(r_1))\right.
\end{aligned} \tag{3.36}
$$

$$-\frac{1}{r}\int_r^{r_1}(m_1-m_2)(s)ds+\frac{1}{r}\int_r^{r_1}(B_1-B_2)(s)ds\Bigg|$$
$$< C\|B_1-B_2\|_{C([r_0,r_1])}+C\left(\frac{|m_1-m_2|(r)}{(r_1-r)^{1/2}}+\frac{|m_1-m_2|(\eta)}{(r_1-\eta)^{1/2}}\right),\qquad \eta\in[r,r_1],$$

where we used (3.12), (3.27), (3.33), (3.34), the mean-value theorem of differentials, the mean-value theorem of integrals, and the equality

$$\tilde{E}_i(r)=\frac{r_1}{r}\tilde{E}_i(r_1)-\frac{1}{r}\int_r^{r_1}(m_i-B_i)(s)ds. \tag{3.37}$$

Putting (3.35) and (3.36) into (3.32), near $r=r_1$, we have

$$\begin{aligned}&(r_1-r)^{1/2}|(m_1-m_2)_r|(r)\\&\quad< C\|B_1-B_2\|_{C([r_0,r_1])}+C\left(\frac{|m_1-m_2|(r)}{(r_1-r)^{1/2}}+\frac{|m_1-m_2|(\eta)}{(r_1-\eta)^{1/2}}\right),\qquad \eta\in[r,r_1].\end{aligned} \tag{3.38}$$

For $\frac{|m_1-m_2|(r)}{(r_1-r)^{1/2}}+\frac{|m_1-m_2|(\eta)}{(r_1-\eta)^{1/2}}$, we claim that there exist two positive constants $\delta_2\in(0,(r_1-r_0)/2)$ and M_2 such that

$$\frac{|m_1-m_2|(r)}{(r_1-r)^{1/2}}\le M_2\|B_1-B_2\|_{C([r_0,r_1])},\quad r\in(r_1-\delta_2,r_1]. \tag{3.39}$$

Otherwise, for any $\delta\in(0,(r_1-r_0)/2)$ and $M>0$, there exists $r_\delta\in(r_1-\delta,r_1]$ such that

$$\frac{|m_1-m_2|(r_\delta)}{(r_1-r_\delta)^{1/2}}>M\|B_1-B_2\|_{C([r_0,r_1])}.$$

Owing to the arbitrariness of δ, we choose $\delta=(r_1-r_0)/n$ with $n=3,4,5,\ldots$, and for arbitrary $M>0$, there is $r_n\in(r_1-\delta,r_1]$ such that

$$\frac{|m_1-m_2|(r_n)}{(r_1-r_n)^{1/2}}>M\|B_1-B_2\|_{C([r_0,r_1])},$$

which indicates that

$$\varliminf_{r_n\to r_1^-}\frac{|m_1-m_2|(r_n)}{(r_1-r_n)^{1/2}}\ge M\|B_1-B_2\|_{C([r_0,r_1])}. \tag{3.40}$$

We can calculate $\lim_{r\to r_1^-}\frac{|m_1-m_2|(r)}{(r_1-r)^{1/2}}$ by applying Lemma 3.2, the L'Hospital rule, and Lemma 3.6,

$$\begin{aligned}&\lim_{r\to r_1^-}\frac{|m_1-m_2|(r)}{(r_1-r)^{1/2}}\\&\quad=\lim_{r\to r_1^-}\frac{m_1(r)-\mathcal{J}}{(r_1-r)^{1/2}}-\lim_{r\to r_1^-}\frac{m_2(r)-\mathcal{J}}{(r_1-r)^{1/2}}\\&\quad=2\lim_{r\to r_1^-}(r_1-r)^{1/2}(m_2)_r(r)-2\lim_{r\to r_1^-}(r_1-r)^{1/2}(m_1)_r(r)\end{aligned} \tag{3.41}$$

$$
\begin{aligned}
&= \mathcal{J}\left(\sqrt{\frac{1}{r_1}\int_{r_0}^{r_1}\left(B_1 - m_1 + \frac{1}{\tau}\right)dr} - \sqrt{\frac{1}{r_1}\int_{r_0}^{r_1}\left(B_2 - m_2 + \frac{1}{\tau}\right)dr}\right)\\
&= \frac{\mathcal{J}}{r_1}\frac{\int_{r_0}^{r_1}(B_1-B_2)(r)dr - \int_{r_0}^{r_1}(m_1-m_2)(r)dr}{\sqrt{\frac{1}{r_1}\int_{r_0}^{r_1}\left(B_1 - m_1 + \frac{1}{\tau}\right)dr} + \sqrt{\frac{1}{r_1}\int_{r_0}^{r_1}\left(B_2 - m_2 + \frac{1}{\tau}\right)dr}}\\
&\le \frac{\mathcal{J}}{r_1}\frac{\int_{r_0}^{r_1}(B_1-B_2)(r)dr}{\sqrt{\frac{1}{r_1}\int_{r_0}^{r_1}\left(B_1 - m_1 + \frac{1}{\tau}\right)dr} + \sqrt{\frac{1}{r_1}\int_{r_0}^{r_1}\left(B_2 - m_2 + \frac{1}{\tau}\right)dr}}\\
&\le C_2\|B_1 - B_2\|_{C([r_0,r_1])}.
\end{aligned}
$$

Note that the constant M in (3.40) is arbitrary, we choose $M = 2C_2$, which contradicts (3.41).

It then follows from (3.38) and (3.39) that

$$
(r_1 - r)^{1/2}|(m_1 - m_2)_r|(r) < C\|B_1 - B_2\|_{C([r_0,r_1])}, \quad r \in (r_1 - \delta_2, r_1]. \tag{3.42}
$$

Recalling the process of calculating $|I_2|$, we have

$$
\begin{aligned}
|\tilde{E}_1 - \tilde{E}_2|(r) &< C\left(\|B_1 - B_2\|_{C([r_0,r_1])} + \frac{|m_1 - m_2|(\eta)}{(r_1-\eta)^{1/2}}\right), \qquad \eta \in [r, r_1],\\
&\le C\|B_1 - B_2\|_{C([r_0,r_1])}, \qquad r \in (r_1 - \delta_2, r_1].
\end{aligned} \tag{3.43}
$$

Besides, by (3.25), (3.39), and (3.43), one has

$$
\begin{aligned}
|(\tilde{E}_1 - \tilde{E}_2)_r|(r) &\le \frac{1}{r}|\tilde{E}_1 - \tilde{E}_2|(r) + \frac{1}{r}|m_1 - m_2|(r) + \frac{1}{r}|B_1 - B_2|(r)\\
&< C\|B_1 - B_2\|_{C([r_0,r_1])}, \qquad r \in (r_1 - \delta_2, r_1].
\end{aligned} \tag{3.44}
$$

Hence, (3.39) together with (3.42)–(3.44) implies the desired estimate (3.31). ∎

Finally, we establish the structural stability estimate of interior subsonic solutions on the middle domain. Until now, we have established the structural stability estimate of solutions on an intrinsic small domain $[r_0, r_0 + \delta_1)$ and the weighed structural stability estimate of solutions on an intrinsic small domain $(r_1 - \delta_2, r_1]$. These ensure us to establish the structural stability estimate of solutions on a certain domain $[r_0 + \delta_0, r_1 - \delta_0]$, where $\delta_0 = \min\{\delta_1, \delta_2\}$.

LEMMA 3.8. *Let all assumptions in Theorem* 3.1 *hold, and let* $B_1(r) \ge B_2(r)$ *on* $[r_0, r_1]$. *Let us take* $\delta_0 = \min\{\delta_1, \delta_2\}$, *then there exists a positive constant* C *independent of* $\|B_1 - B_2\|_{C([r_0,r_1])}$ *such that*

$$
\|m_1 - m_2\|_{C^1([r_0+\delta_0, r_1-\delta_0])} + \|\tilde{E}_1 - \tilde{E}_2\|_{C^1([r_0+\delta_0, r_1-\delta_0])} < C\|B_1 - B_2\|_{C([r_0,r_1])}. \tag{3.45}
$$

Proof. From (2.2) and Lemma 3.2, we get the boundedness of $m_1(r)$ and $m_2(r)$ on $[r_0 + \delta_0, r_1 - \delta_0]$,

$$
\mathcal{J} < \kappa := \mathcal{J} + \lambda_2 \sin\left(\pi \cdot \frac{\delta_0}{r_1 - r_0}\right) \le m_2(r) \le m_1(r) \le \overline{B}_1 + \frac{1}{\tau}. \tag{3.46}
$$

Taking the difference of $(3.13)|_{i=1}$ and $(3.13)|_{i=2}$ and using the mean-value theorem of differentials, we obtain

$$(3.47)\quad \begin{aligned}&(m_1-m_2)_r\\ &=\frac{(m_1)^3}{(m_1)^2-\mathcal{J}^2}\left(\tilde{E}_1+\frac{1}{r}-\frac{\mathcal{J}}{m_1\tau}\right)-\frac{(m_2)^3}{(m_2)^2-\mathcal{J}^2}\left(\tilde{E}_2+\frac{1}{r}-\frac{\mathcal{J}}{m_2\tau}\right)\\ &=\frac{(m_1)^3}{(m_1)^2-\mathcal{J}^2}\left(\tilde{E}_1+\frac{1}{r}\right)-\frac{\mathcal{J}}{\tau}\frac{(m_1)^2}{(m_1)^2-\mathcal{J}^2}\\ &\quad-\frac{(m_2)^3}{(m_2)^2-\mathcal{J}^2}\left(\tilde{E}_2+\frac{1}{r}\right)+\frac{\mathcal{J}}{\tau}\frac{(m_2)^2}{(m_2)^2-\mathcal{J}^2}\\ &=\left(\tilde{E}_1+\frac{1}{r}\right)\left(\frac{(m_1)^3}{(m_1)^2-\mathcal{J}^2}-\frac{(m_2)^3}{(m_2)^2-\mathcal{J}^2}\right)+\frac{(m_2)^3}{(m_2)^2-\mathcal{J}^2}(\tilde{E}_1-\tilde{E}_2)\\ &\quad-\frac{\mathcal{J}}{\tau}\left(\frac{(m_1)^2}{(m_1)^2-\mathcal{J}^2}-\frac{(m_2)^2}{(m_2)^2-\mathcal{J}^2}\right)\\ &=\left(\left(\tilde{E}_1+\frac{1}{r}\right)g_1'(\zeta_1)-\frac{\mathcal{J}}{\tau}g_2'(\zeta_2)\right)(m_1-m_2)\\ &\quad+\frac{(m_2)^3}{(m_2)^2-\mathcal{J}^2}(\tilde{E}_1-\tilde{E}_2),\quad \zeta_1,\zeta_2\in(m_2,m_1),\end{aligned}$$

where

$$g_1(s)=\frac{s^3}{s^2-\mathcal{J}^2},\qquad g_2(s)=\frac{s^2}{s^2-\mathcal{J}^2}.$$

We multiply (3.47) by $(m_1-m_2)(r)$ and use (3.33), (3.46), and Cauchy's inequality to compute that

$$(3.48)\quad ((m_1-m_2)^2)_r(r)<C(m_1-m_2)^2(r)+C(\tilde{E}_1-\tilde{E}_2)^2(r),\quad r\in[r_0+\delta_0,r_1-\delta_0].$$

Multiplying (3.25) by $(\tilde{E}_1-\tilde{E}_2)(r)$ and using Cauchy's inequality, we have

$$(3.49)\quad \begin{aligned}((\tilde{E}_1-\tilde{E}_2)^2)_r(r)&<C(m_1-m_2)^2(r)+C(\tilde{E}_1-\tilde{E}_2)^2(r)\\ &\quad+\|B_1-B_2\|^2_{C([r_0,r_1])},\quad r\in[r_0+\delta_0,r_1-\delta_0].\end{aligned}$$

Adding (3.49) to (3.48) yields

$$(3.50)\quad \begin{aligned}&((m_1-m_2)^2+(\tilde{E}_1-\tilde{E}_2)^2)_r(r)\\ &\quad<C((m_1-m_2)^2+(\tilde{E}_1-\tilde{E}_2)^2)(r)\\ &\qquad+\|B_1-B_2\|^2_{C([r_0,r_1])},\quad r\in[r_0+\delta_0,r_1-\delta_0],\end{aligned}$$

then we get

$$(3.51)\quad \begin{aligned}&((m_1-m_2)^2+(\tilde{E}_1-\tilde{E}_2)^2)(r)\\ &\quad<C((m_1-m_2)^2+(\tilde{E}_1-\tilde{E}_2)^2)(r_0+\delta_0)\\ &\qquad+C\|B_1-B_2\|^2_{C([r_0,r_1])},\quad r\in[r_0+\delta_0,r_1-\delta_0].\end{aligned}$$

Since $\delta_0=\min\{\delta_1,\delta_2\}$ and the functions $(m_1-m_2,\tilde{E}_1-\tilde{E}_2)$ are continuous at $r=r_0+\delta_1$, we can derive from Lemma 3.4 that

$$(3.52)\quad ((m_1-m_2)^2+(\tilde{E}_1-\tilde{E}_2)^2)(r_0+\delta_0)\le C\|B_1-B_2\|^2_{C([r_0,r_1])},$$

which in combination with (3.51) leads to

$$|m_1-m_2|(r)+|\tilde{E}_1-\tilde{E}_2|(r)<C\|B_1-B_2\|_{C([r_0,r_1])},\quad r\in[r_0+\delta_0,r_1-\delta_0]. \tag{3.53}$$

From (3.25), (3.47), and (3.53), it is easy to see that

$$|(m_1-m_2)_r|(r)+|(\tilde{E}_1-\tilde{E}_2)_r|(r)<C\|B_1-B_2\|_{C([r_0,r_1])},\quad r\in[r_0+\delta_0,r_1-\delta_0]. \tag{3.54}$$

Hence, (3.53)–(3.54) imply the desired estimate (3.45). ∎

Now we prove the estimates (3.33) and (3.34) in the following Lemma.

LEMMA 3.9. *Let all assumptions in Theorem* 3.1 *be satisfied, and let* $B_1(r)\geq B_2(r)$ *on* $[r_0,r_1]$. *Then,*

$$|\tilde{E}_i(r)|<C(\overline{B}_1+1/\tau),\quad r\in[r_0,r_1],\quad i=1,2, \tag{3.55}$$

$$|\tilde{E}_1(r_1)-\tilde{E}_2(r_1)|<C\|B_1-B_2\|_{C([r_0,r_1])}, \tag{3.56}$$

where C *is a positive constant independent of* $\|B_1-B_2\|_{C([r_0,r_1])}$.

Proof. By (3.23) and (3.12), we obtain, for $i=1,2$,

$$\begin{aligned}|\tilde{E}_i(r)|&\leq\left|\frac{1}{\tau}-\frac{1}{r_0}\right|+\frac{1}{r_0}\int_{r_0}^{r_1}(m_i+B_i)(s)ds\\&<C(\overline{B}_1+1/\tau),\qquad r\in[r_0,r_1].\end{aligned} \tag{3.57}$$

Taking $r=r_1$ in (3.23) yields

$$\tilde{E}_i(r_1)=\frac{r_0}{r_1}\left(\frac{1}{\tau}-\frac{1}{r_0}\right)+\frac{1}{r_1}\int_{r_0}^{r_1}(m_i-B_i)(s)ds, \tag{3.58}$$

then, by the mean-value theorem of integrals, we have

$$\begin{aligned}|\tilde{E}_1(r_1)-\tilde{E}_2(r_1)|&=\frac{1}{r_1}\left|\int_{r_0}^{r_1}(m_1-m_2)(s)ds-\int_{r_0}^{r_1}(B_1-B_2)(s)ds\right|\\&\leq\frac{1}{r_1}\int_{r_0}^{r_1}|m_1-m_2|(s)ds+C\|B_1-B_2\|_{C([r_0,r_1])}\\&\leq C|m_1-m_2|(\theta)+C\|B_1-B_2\|_{C([r_0,r_1])},\qquad\theta\in[r_0,r_1].\end{aligned} \tag{3.59}$$

It follows from (3.22), (3.39), and (3.53) that there exits a positive constant C independent of $\|B_1-B_2\|_{C([r_0,r_1])}$ such that

$$|m_1-m_2|(\theta)<C\|B_1-B_2\|_{C([r_0,r_1])}, \tag{3.60}$$

wherever the point θ is located in the whole domain $[r_0,r_1]$. Thus, (3.59) and (3.60) imply (3.56). ∎

Based on Lemmas 3.4, 3.7, and 3.8, we can derive the following proposition.

PROPOSITION 3.10. *Suppose that the conditions in Theorem* 3.1 *hold, and let* $B_1(r)\geq B_2(r)$ *on* $[r_0,r_1]$. *Then, the interior subsonic solutions* $(m_i,\tilde{E}_i)(r)$ *to* (3.1) *are structurally stable in the sense that*

$$\begin{aligned}&\|m_1-m_2\|_{C([r_0,r_1])}+\|(r_1-r)^{1/2}(m_1-m_2)_r\|_{C([r_0,r_1])}\\&\quad+\|\tilde{E}_1-\tilde{E}_2\|_{C^1([r_0,r_1])}<C\|B_1-B_2\|_{C([r_0,r_1])},\end{aligned}$$

where C *is a positive constant independent of* $\|B_1-B_2\|_{C([r_0,r_1])}$.

Proof of Theorem 3.1. Let $\bar{B}(r) = \max\{B_1(r), B_2(r)\}$ and $(\bar{m}(r), \bar{E}(r))$ be the corresponding solution to the problem (3.1). Then the comparison principle shows, for any $r \in [r_0, r_1]$,

$$m_1(r) \le \bar{m}(r), \qquad m_2(r) \le \bar{m}(r).$$

Thanks to Proposition 3.10, we have

$$\begin{aligned}
&\|m_1 - m_2\|_{C([r_0,r_1])} + \|(r_1 - r)^{1/2}(m_1 - m_2)_r\|_{C([r_0,r_1])} + \|\tilde{E}_1 - \tilde{E}_2\|_{C^1([r_0,r_1])} \\
&\le \|m_1 - \bar{m}\|_{C([r_0,r_1])} + \|(r_1 - r)^{1/2}(m_1 - \bar{m})_r\|_{C([r_0,r_1])} + \|\tilde{E}_1 - \bar{E}\|_{C^1([r_0,r_1])} \\
&\quad + \|\bar{m} - m_2\|_{C([r_0,r_1])} + \|(r_1 - r)^{1/2}(\bar{m} - m_2)_r\|_{C([r_0,r_1])} + \|\bar{E} - \tilde{E}_2\|_{C^1([r_0,r_1])} \\
&< C\|\bar{B} - B_1\|_{C([r_0,r_1])} + \|\bar{B} - B_2\|_{C([r_0,r_1])} \\
&= C\|B_1 - B_2\|_{C([r_0,r_1])}.
\end{aligned}$$

The proof of Theorem 3.1 is complete. ∎

Next, we state the structural stability of interior subsonic solutions to the system (1.10) in the 3-D case. Analogously to Theorem 3.1, and in a similar manner, we can obtain the same result about the interior subsonic solutions $(m_i, \tilde{E}_i)(r)$ to the following system:

$$\begin{cases} \left(1 - \dfrac{\mathcal{J}^2}{(m_i)^2}\right)(m_i)_r = m_i\left(\tilde{E}_i + \dfrac{2}{r}\right) - \dfrac{\mathcal{J}}{\tau}, \\ (r^2\tilde{E}_i)_r = m_i - B_i(r), \qquad\qquad r \in (r_0, r_1), \quad i = 1, 2, \\ m_i(r_0) = m_i(r_1) = \mathcal{J}. \end{cases} \tag{3.61}$$

THEOREM 3.11. *For $i = 1, 2$, let doping profiles $B_i(r) \in C\big([r_0, r_1]\big)$, $\mathcal{J} + 2 < \underline{B}_i \le B_i(r) \le \overline{B}_i$ on $[r_0, r_1]$, and τ satisfy*

$$0 < \tau < \min\left\{\frac{-r_0 + r_0\sqrt{1 + (B_1(r_0) - \mathcal{J} - 2)/8}}{B_1(r_0) - \mathcal{J} - 2}, \frac{-r_0 + r_0\sqrt{1 + (B_2(r_0) - \mathcal{J} - 2)/8}}{B_2(r_0) - \mathcal{J} - 2}, \frac{r_0}{2}\right\}. \tag{3.62}$$

Then, the interior subsonic solutions $(m_i, \tilde{E}_i)(r)$ to (3.61) *relative to doping profiles $B_i(r)$ are structurally stable in the sense that*

$$\begin{aligned} &\|m_1 - m_2\|_{C([r_0,r_1])} + \|(r_1 - r)^{1/2}(m_1 - m_2)_r\|_{C([r_0,r_1])} \\ &\quad + \|\tilde{E}_1 - \tilde{E}_2\|_{C^1([r_0,r_1])} < C\|B_1 - B_2\|_{C([r_0,r_1])}, \end{aligned} \tag{3.63}$$

where $C > 0$ is a constant independent of $\|B_1 - B_2\|_{C([r_0,r_1])}$.

Here we only point out the differences with Lemmas 3.3 and 3.6, by the alike processes as that in Lemmas 3.3 and 3.6, we can get the following lemma.

LEMMA 3.12. *For $i = 1, 2$, suppose that doping profiles $B_i(r)$ and τ satisfy conditions in Theorem* 3.11. *Then,*

$$\lim_{r\to r_0^+}(m_i)_r(r) = \frac{\mathcal{J}}{4}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - \frac{8}{r_0^2\tau}((B_i(r_0) - \mathcal{J} - 2)\tau + 2r_0)}\right) > 0, \tag{3.64}$$

$$\lim_{r\to r_1^-}(r_1 - r)^{1/2}(m_i)_r(r) = -\frac{\mathcal{J}}{2r_1}\sqrt{\int_{r_0}^{r_1}\left(B_i - m_i - 2 + \frac{r_1 + r_0}{\tau}\right)(s)ds} < 0. \tag{3.65}$$

Proof of Theorem 2.3. Combining Theorems 3.1 and 3.11, we can immediately obtain Theorem 2.3.

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REFERENCES

[1] P. Amster, M. P. B. Varela, A. Jüngel, and M. C. Mariani, *Subsonic solutions to a one-dimensional non-isentropic hydrodynamic model for semiconductors*, J. Math. Anal. Appl., 258 (2001), pp. 52–62.

[2] U. M. Ascher, P. A. Markowich, P. Pietra, and C. Schmeiser, *A phase plane analysis of transonic solutions for the hydrodynamic semiconductor model*, Math. Models Methods Appl. Sci., 1 (1991), pp. 347–376.

[3] M. Bae, B. Duan, J. Xiao, and C. J. Xie, *Structural stability of supersonic solutions to the Euler-Poisson system*, Arch. Ration. Mech. Anal., 239 (2021), pp. 679–731.

[4] M. Bae, B. Duan, and C. J. Xie, *Subsonic solutions for steady Euler–Poisson system in two-dimensional nozzles*, SIAM J. Math. Anal., 46 (2014), pp. 3455–3480, https://doi.org/10.1137/13094222X.

[5] M. Bae, B. Duan, and C. J. Xie, *Subsonic flow for the multidimensional Euler-Poisson system*, Arch. Ration. Mech. Anal., 220 (2016), pp. 155–191.

[6] M. Bae and H. Park, *Three-dimensional supersonic flows of Euler-Poisson system for potential flow*, Commun. Pure Appl. Anal., 20 (2021), pp. 2421–2440.

[7] M. Bae and Y. Park, *Radial transonic shock solutions of Euler-Poisson system in convergent nozzles*, Discrete Contin. Dyn. Syst. Ser. S, 11 (2018), pp. 773–791.

[8] K. Bløtekjær, *Transport equations for electrons in two-valley semiconductors*, IEEE Trans. Electron Devices, 17 (1970), pp. 38–47.

[9] L. Chen, D. Li, M. Mei, and G. J. Zhang, *Quasi-neutral limit to steady-state hydrodynamic model of semiconductors with degenerate boundary*, SIAM J. Math. Anal., 55 (2023), pp. 2813–2837, https://doi.org/10.1137/22M1504135.

[10] L. Chen, M. Mei, G. J. Zhang, and K. J. Zhang, *Steady hydrodynamic model of semiconductors with sonic boundary and transonic doping profile*, J. Differential Equations, 269 (2020), pp. 8173–8211.

[11] L. Chen, M. Mei, G. J. Zhang, and K. J. Zhang, *Radial solutions of hydrodynamic model of semiconductors with sonic boundary*, J. Math. Anal. Appl., 501 (2021), pp. 125–187.

[12] L. Chen, M. Mei, G. J. Zhang, and K. J. Zhang, *Transonic steady-states of Euler–poisson equations for semiconductor models with sonic boundary*, SIAM J. Math. Anal., 54 (2022), pp. 363–388, https://doi.org/10.1137/21M1416369.

[13] P. Degond and P. A. Markowich, *On a one-dimensional steady-state hydrodynamic model for semiconductors*, Appl. Math. Lett., 3 (1990), pp. 25–29.

[14] P. Degond and P. A. Markowich, *A steady state potential flow model for semiconductors*, Ann. Mat. Pura Appl., 165 (1993), pp. 87–98.

[15] Y. H. Feng, H. F. Hu, and M. Mei, *Structural Stability of Interior Subsonic Steady-states to Hydrodynamic Model for Semiconductors with Sonic Boundary*, preprint, https://arxiv.org/abs/2204.01236, 2022.

[16] Y. H. Feng, M. Mei, and G. J. Zhang, *Nonlinear structural stability and linear dynamic instability of transonic steady-states to a hydrodynamic model for semiconductors*, J. Differential Equations, 344 (2023), pp. 131–171.

[17] I. M. Gamba, *Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductors*, Comm. Partial Differential Equations, 17 (1992), pp. 553–577.

[18] I. M. Gamba and C. S. Morawetz, *A viscous approximation for a 2-d steady semiconductor or transonic gas dynamic flow: Existence theorem for potential flow*, Comm. Pure Appl. Math., 49 (1996), pp. 999–1049.

[19] A. Jüngel, *Quasi-Hydrodynamic Semiconductor Equations*, Progr. Nonlinear Differential Equations Appl., Vol. 41, Birkhäuser Verlag, Basel, 2001.

[20] J. Y. Li, M. Mei, G. J. Zhang, and K. J. Zhang, *Steady hydrodynamic model of semiconductors with sonic boundary:* (i) *subsonic doping profile*, SIAM J. Math. Anal., 49 (2017), pp. 4767–4811, https://doi.org/10.1137/17M1127235.

[21] J. Y. Li, M. Mei, G. J. Zhang, and K. J. Zhang, *Steady hydrodynamic model of semiconductors with sonic boundary:* (II) *supersonic doping profile*, SIAM J. Math. Anal., 50 (2018), pp. 718–734, https://doi.org/10.1137/17M1129477.

[22] T. Luo, J. Rauch, C. J. Xie, and Z. P. Xin, *Stability of transonic shock solutions for one-dimensional Euler-Poisson equations*, Arch. Ration. Mech. Anal., 202 (2011), pp. 787–827.

[23] T. Luo and Z. P. Xin, *Transonic shock solutions for a system of euler-poisson equations*, Commun. Math. Sci., 10 (2012), pp. 419–462.

[24] P. A. Markowich, *On steady state Euler-Poisson models for semiconductors*, Z. Angew. Math. Phys., 42 (1991), pp. 389–407.

[25] P. A. Markowich, C. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag, Vienna, 1990.

[26] S. Nishibata, M. Ohnawa, and M. Suzuki, *Asymptotic stability of boundary layers to the Euler–Poisson equations arising in plasma physics*, SIAM J. Math. Anal., 44 (2012), pp. 761–790.

[27] S. Nishibata and M. Suzuki, *Asymptotic stability of a stationary solution to a thermal hydrodynamic model for semiconductors*, Arch. Ration. Mech. Anal., 192 (2009), pp. 187–215.

[28] S. Nishibata and M. Suzuki, *Hierarchy of Semiconductor Equations: Relaxation Limits with Initial Layers for Large Initial Data*, MSJ Memoirs, Vol. 26, Mathematical Society of Japan, Tokyo, 2011.

[29] Y. Peng and I. Violet, *Example of supersonic solutions to a steady state Euler-Poisson system*, Appl. Math. Lett., 19 (2006), pp. 1335–1340.

[30] M. D. Rosini, *A phase analysis of transonic solutions for the hydrodynamic semiconductor model*, Quart. Appl. Math., 63 (2005), pp. 251–268.

[31] R. Schreiber, *Regularity of singular two-point boundary value problems*, SIAM J. Math. Anal., 12 (1981), pp. 104–109, https://doi.org/10.1137/0512010.

[32] C. Wang, *Global smooth sonic-supersonic flows in a class of critical nozzles*, SIAM J. Math. Anal., 54 (2022), pp. 1820–1859, https://doi.org/10.1137/20M1373906.

[33] C. Wang and Z. Xin, *On sonic curves of smooth subsonic-sonic and transonic flows*, SIAM J. Math. Anal., 48 (2016), pp. 2414–2453, https://doi.org/10.1137/16M1056407.

[34] C. Wang and Z. Xin, *Smooth transonic flows of meyer type in de laval nozzles*, Arch. Ration. Mech. Anal., 232 (2019), pp. 1597–1647.

[35] C. Wang and Z. Xin, *Regular subsonic-sonic flows in general nozzles*, Adv. Math., 380 (2021), pp. 1–56.

[36] M. Wei, M. Mei, G. Zhang, and K. Zhang, *Smooth transonic steady-states of hydrodynamic model for semiconductors*, SIAM J. Math. Anal., 53 (2021), pp. 4908–4932, https://doi.org/10.1137/21M139582X.

[37] S. Weng, Z. Xin, and H. Yuan, *On some smooth symmetric transonic flows with nonzero angular velocity and vorticity*, Math. Models Methods Appl. Sci., 31 (2021), pp. 2773–2817.